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A METHOD OF SOLVING LINEAR DIFFERENTIAL EQUATIONS. SECOND PAPER.

BY P. A. LAMBERT.

The object of this paper is to apply to linear partial differential equations the method of solution applied to ordinary linear differential equations in the paper entitled "A Method of Solving Linear Differential Equations" published in the *Annals of Mathematics*, July, 1910.

Let the given differential equation be

$$(1)\qquad f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \cdots \frac{\partial^n u}{\partial y^n}\right) = 0.$$

The method of solution proposed consists of the following steps:

(*a*) Break up the function f into two parts, one of which, f_1, equated to zero gives a differential equation which may be readily solved, and introduce a parameter t as a factor of the second part f_2, so that the given equation, $f_1 + f_2 = 0$, is replaced by

$$(2)\qquad f_1 + tf_2 = 0.$$

(*b*) Assume that the series

$$(3)\qquad u = u_0 + u_1 t + u_2 t^2 + u_3 t^3 + \cdots,$$

where $u_0, u_1, u_2, u_3, \cdots$ are undetermined functions of x and y, makes equation (2) an identity. Substitute the expression (3) in equation (2) and determine these functions by solving the differential equations formed by equating to zero the coefficients of successive powers of t in the resulting identity.

(*c*) Substitute these values of $u_0, u_1, u_2, u_3, \cdots$ in (3), and replace t by unity. Then see if

$$(4)\qquad u = u_0 + u_1 + u_2 + u_3 + \cdots$$

is convergent and satisfies equation (1).

This method of solving differential equations will be called the parametric method.

The results obtained by the application of this method are not new. The actual solution of an ordinary linear differential equation by the

parametric method is simpler in theory and decidedly less laborious than by the method which assumes the solution to be

$$y = \sum_{r=0}^{r=\infty} A_r x^{m+rs}$$

and requires the determination of the constants A_r, m and s.

The method of solution of linear ordinary differential equations outlined by Schlesinger* is practically identical with a special application of the parametric method. In the application of the method outlined by Schlesinger, which was established by Caqué† by studying equations of finite differences, the given differential equation,

$$f\left(x, y, \frac{dy}{dx}, \cdots \frac{d^n y}{dx^n}\right) = 0$$

is broken up into

$$f_1 - f_2 = p(x) \quad \text{or} \quad D_1(y) - D_2(y) = p(x),$$

where D_1 must contain the derivative of highest order. The solution of $D_1(y) = 0$ is called u_0 and the solution of the given equation is written $y = u_0 + u$, so that

$$D_1(u_0 + u) = D_2(u_0 + u) + p(x),$$

and u must be the principal integral of

$$D_1(u) - D_2(u) = F_0(x),$$

where $F_0(x)$ represents the known function

$$p(x) + D_2(u_0).$$

If u_1 is the principal integral of

$$D_1(u) = F_0(x)$$

and $u = u_1 + v$, v must be the principal integral of

$$D_1(v) = D_2(v) + F_1(x)$$

where

$$F_1(x) = D_2(u_1).$$

Repeated application of this process gives

$$y = u_0 + u_1 + u_2 + u_3 + \cdots,$$

*Handbuch der Theorie der linearen Differentialgleichungen, pp. 370–377.
†Liouville's Journal, Series II, Vol. 9, p. 185.

which is proved to be convergent and a solution of the given differential equation.

In applying the parametric method the given differential equation is replaced by

$$f_1 + tf_2 = 0,$$

where f_1 is selected so that (1) the equation $f_1 = 0$ can be integrated and (2) the resulting series shall be convergent. It is frequently advantageous to select f_1 so that it will not contain the highest derivative of the given equation.

Schwarz* discusses the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + au = 0,$$

and Darboux† discusses the partial differential equation

$$\frac{\partial^2 u}{\partial x\,\partial y} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu,$$

where a, b, c, are functions of x and y, by a method of successive approximations which is essentially the same as the parametric method. However, the use of the parameter, which characterizes my method of solution, does not seem to occur in the literature of differential equations. Moreover, the parametric method seems better adapted to the actual determination of the solution of the partial differential equation than the method of successive approximations as used by Schwarz and Darboux.

The parametric method will be exemplified by applying it to several examples.

In Example I the method is applied to a first order equation to throw the method into prominence. Example II is the second order differential equation of fundamental importance in mathematical physics and the theory of functions. The solution containing two arbitrary functions is found. Examples III, IV and V show how to find particular integrals under different conditions

The differential equations of the first five examples have constant coefficients. In Example VI, the differential equation of Euler and Poisson,‡ of importance in mathematical physics and differential geometry, the coefficients are functions of x and y.

*Abhandlungen, Vol. I, pp. 241–265.

† Théorie générale des surfaces, vol. IV, pp. 353–367.

‡ Darboux, Théorie des surfaces, vol. II, p. 54.

Example I.
$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=nu.$$

Replace the equation by

$$\frac{\partial u}{\partial y}-nu+\frac{\partial u}{\partial x}t=0$$

and assume

$$u=u_0+u_1t+u_2t^2+u_3t^3+\cdots.$$

There results

$$\left.\begin{array}{c}\dfrac{\partial u_0}{\partial y}\\[2mm] -nu_0\end{array}\right|\left.\begin{array}{c}+\dfrac{\partial u_1}{\partial y}\\[2mm] -nu_1\\[2mm] +\dfrac{\partial u_0}{\partial x}\end{array}\right|t+\left.\begin{array}{c}\dfrac{\partial u_2}{\partial y}\\[2mm] -nu_2\\[2mm] +\dfrac{\partial u_1}{\partial x}\end{array}\right|t^2+\cdots\equiv 0.$$

Solving the differential equations obtained by equating to zero the coefficients of powers of t in this identity,

$$u_0=e^{ny}\varphi(x),$$

$$u_1=-e^{ny}\varphi'(x)y,$$

$$u_2=e^{ny}\varphi''(x)\frac{y^2}{2!},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Substituting in the value of u and making t unity,

$$u=e^{ny}\left[\varphi(x)-\varphi'(x)y+\varphi''(x)\frac{y^2}{2!}-\cdots\right],$$

whence by Taylor's series

$$u=e^{ny}\varphi(x-y),$$

the integral of the given differential equation containing an arbitrary function.

Example II.
$$\frac{\partial^2 u}{\partial x^2}+a^2\frac{\partial^2 u}{\partial y^2}=0.$$

Replacing this equation by

$$\frac{\partial^2 u}{\partial x^2}+a^2t\frac{\partial^2 u}{\partial y^2}=0,$$

and assuming

$$u=u_0+u_1t+u_2t^2+u_3t^3+\cdots,$$

there results

$$\begin{array}{c|c|c}
\dfrac{\partial^2 u_0}{\partial x^2} + & \dfrac{\partial^2 u_1}{\partial x^2}\, t + & \dfrac{\partial^2 u_2}{\partial x^2}\, t^2 + \cdots \equiv 0. \\
+ a^2 \dfrac{\partial^2 u_0}{\partial y_2} & + a^2 \dfrac{\partial^2 u_1}{\partial y^2} &
\end{array}$$

From the series of differential equations obtained by equating to zero the coefficients of powers of t in this identity may be found

$$u_0 = (A + Bx)\varphi(y),$$

$$u_1 = -a^2\left(A\frac{x^2}{2!} + B\frac{x^3}{3!}\right)\varphi''(y),$$

$$u_2 = a^4\left(A\frac{x^4}{4!} + B\frac{x^5}{5!}\right)\varphi'''(y),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Substituting in u and making t unity,

$$u = A\varphi(y) - A\frac{a^2x^2}{2!}\varphi''(y) + A\frac{a^4x^4}{4!}\varphi^{\mathrm{IV}}(y) + \cdots$$

$$+ B\varphi(y)x - B\frac{a^2x^3}{3!}\varphi''(y) + B\frac{a^4x^5}{5!}\varphi^{\mathrm{IV}}(y) - \cdots.$$

Now if $\varphi(y)$ is made e^y in order that $\varphi(y)$ and all its derivatives shall have the same value, and if the arbitrary constant B is replaced by Ca, it is readily seen that

$$u = Ae^y \cos (ax) + Ce^y \sin (ax).$$

It is evident that this solution of the differential equation may be written in the form

$$u = Ae^{ny}\cos\frac{ax}{n} + Ce^{ny}\sin\frac{ax}{n}.$$

If it is desired that u shall contain all the successive derivatives of $\varphi(y)$ the values of u_0, u_1, u_2, u_3, $\cdots$ may be written

$$u_0 = \varphi(y) + Bx\varphi'(y)$$

$$u_1 = -\varphi''(y)\frac{a^2x^2}{2!} - B\varphi'''(y)\frac{a^2x^3}{3!}$$

$$u_2 = \varphi^{\mathrm{IV}}(y)\frac{a^4x^4}{4!} + B\varphi^{\mathrm{V}}(y)\frac{a^4x^5}{5!}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Whence

$$u = \varphi(y) + \varphi'(y)Bx - \varphi''(y)\frac{a^2x^2}{2!}$$

$$- \varphi'''(y)B\frac{a^2x^3}{3!} + \varphi^{\mathrm{IV}}(y)\frac{a^4x^4}{4!} + \cdots.$$

The terms of n which have been written are the first five terms of the expansion of a function by Taylor's series provided $B = ai$ or $B = -ai$, where $i = \sqrt{-1}$.

The terms of u which have been written now suggest that

$$u = \varphi_1(y + iax) + \varphi_2(y - iax),$$

which on trial is found to be correct.

Of this solution containing two arbitrary functions the former solution containing two arbitrary constants is a special case.

Example III. $$\frac{\partial^3 u}{\partial x^2 \partial y} - 2\frac{\partial^3 y}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial y^3} = \frac{1}{x^2}.$$

Replacing this equation by

$$\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{1}{x^2} - \left(2\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 u}{\partial y^3}\right)t = 0,$$

and applying the method

$$u_0 = \varphi_1(x) + (A + Bx)\varphi_2(y) + (A_1 + B_1x)\varphi_2'(y) - y \log x.$$

By actual trial it is found that the following four values of u_0, parts of this general expression for u_0,

$$u_0 = \varphi_1(x),$$

$$u_0 = \varphi_2(y) + B_1x\varphi_2'(y),$$

$$u_0 = Bx\varphi_3(y),$$

$$u_0 = -y \log x,$$

lead to solutions of the differential equation.

The solution of the equation, the sum of the solutions corresponding to these four values of u_0, is

$$u = \varphi_1(x) + \varphi_2(x + y) + x\varphi_3(x + y) - y \log x.$$

Example IV. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} - 3\dfrac{\partial u}{\partial x} + 3\dfrac{\partial u}{\partial y} = e^{x+2y} + xy.$

The complementary integral is the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 3\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 0.$$

Replacing this equation by

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial u}{\partial x} - \left(\frac{\partial^2 u}{\partial y^2} - 3\frac{\partial u}{\partial y}\right)t = 0,$$

we find $u_0 = e^{3x}\varphi(y)$, which suggests that e^{3x} is a factor of the complementary integral.

Transforming the equation by the relation $u = e^{3x}\cdot v$, a partial differential equation is formed of which

$$v = \varphi(x - y)$$

is a solution. It follows that

$$u = e^{3x}\varphi(x - y)$$

is a complementary integral.

If the differential equation which determines the complementary integral is replaced by

$$\left(\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial u}{\partial x}\right)t - \left(\frac{\partial^2 u}{\partial y^2} - 3\frac{\partial u}{\partial y}\right) = 0$$

it is found in like manner that

$$u = e^{3y}\varphi(x - y)$$

is a complementary integral.

These two complementary integrals are not independent. Replacing the differential equation by

$$\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right)t - 3\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 0,$$

a third and independent complementary integral is found

$$u = \varphi(x + y).$$

The particular integral corresponding to the term xy is found by applying the method to

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial u}{\partial x} - xy - \left(\frac{\partial^2 u}{\partial y^2} - 3\frac{\partial u}{\partial y}\right)t = 0.$$

To find the particular integral corresponding to the term e^{x+2y} transform the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 3\frac{du}{\partial x} + 3\frac{\partial u}{\partial y} = e^{x+2y}$$

by the relation

$$u = v\cdot e^{x+2y}.$$

The equation in v is

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} = 1.$$

A special solution of this equation is $v = -y$, and the corresponding particular integral of the given equation is

$$u = -y\cdot e^{x+2y}.$$

The solution of the given differential equation is the sum of the two independent complementary integrals and the two parts of the particular integral.

Example V. To find a particular integral of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x\partial y} - 6\frac{\partial^2 u}{\partial y^2} = x^2 \sin(x+y).$$

Replace the right-hand member by $x^2 e^{ix+iy}$ and transform the resulting equation by the relation

$$u = e^{ix+iy}\cdot v.$$

The particular integral of the equation in v can be found by the method of example IV.

The particular integral of the given equation is the coefficient of i in the corresponding value of u.

So far the parametric method has been applied to linear partial differential equations with constant coefficients. The following example shows that the method may be applied with advantage to equations with variable coefficients.

Example VI. $$\frac{\partial^2 u}{\partial x\partial y} - \frac{a}{x-y}\frac{\partial u}{\partial x} + \frac{b}{x-y}\frac{\partial u}{\partial y} = 0.$$

Replacing this equation by

$$a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial y} = (x - y)\frac{\partial^2 u}{\partial x \partial y}t$$

and assuming that

$$u = u_0 + u_1 t + u_2 t^2 + u_3 t^3 + \cdots,$$

it follows at once that

$$u_0 = \varphi(ay + bx).$$

Solutions of the given equations which are homogeneous polynomials are found by giving u_0 the successive values

$$u_0 = ay + bx,$$

$$u_0 = (ay + bx)^2,$$

$$u_0 = (ay + bx)^3,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The given equation may be written

$$\left(x\frac{\partial^2 u}{\partial x \partial y} + b\frac{\partial u}{\partial y}\right) - \left(y\frac{\partial^2 u}{\partial x \partial y} + a\frac{\partial u}{\partial x}\right) = 0.$$

A solution of the first part of this equation placed equal to zero is

$$u = x^{-b}\varphi_1(y);$$

a solution of the second part placed equal to zero is

$$u = y^{-a}\varphi_2(x).$$

Hence a solution of the given equation is

$$u = x^{-b}y^{-a},$$

from which we may get the more general solution

$$u = (x + m)^{-b}(y + m)^{-a},$$

where m is an arbitrary constant.

The given differential equation may be replaced by either of the equations

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{a}{x - y}\frac{\partial u}{\partial x} + \frac{b}{x - y}\frac{\partial u}{\partial y}t = 0,$$

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{b}{x - y}\frac{\partial u}{\partial y} - \frac{a}{x - y}\frac{\partial u}{\partial x}t = 0.$$

From the first equation a special value of u_0 is

$$u_0 = \frac{-(x-y)^{1-a}}{1-a}.$$

The use of this value of u_0 suggests that $(x-y)^{1-a}$ is a factor of the solution of the given equation. In like manner from the second equation

$$u_0 = \frac{-(x-y)^{1-b}}{1-b},$$

which suggests that $(x-y)^{1-b}$ is a factor of the solution of the given equation.

From these suggestions it is inferred that $(x-y)^{1-a-b}$ is a factor of the solution of the given equation. This inference is correct, for the relation

$$u = (x-y)^{1-a-b}\cdot v$$

transforms the given equation into

$$\frac{\partial^2 v}{\partial x \partial y} - \frac{1-b}{x-y}\frac{\partial v}{\partial x} + \frac{1-a}{x-y}\frac{\partial v}{\partial y} = 0.$$

By the preceding paragraph a solution of this equation is

$$v = (x+m)^{a-1}(y+n)^{b-1}.$$

The corresponding solution of the given equation is

$$u = (x+m)^{a-1}(y+m)^{b-1}(x-y)^{1-a-b}.$$

LEHIGH UNIVERSITY, February, 1911.

DUALITY IN PROJECTIVE GEOMETRY.

By N. J. Lennes.

Veblen and Young have given a set of independent assumptions for projective geometry.* Their assumptions are stated in terms of the abstract (undefined) symbols *point* and classes of points called *lines*. The plane, three-space and spaces of higher dimensions are defined as classes of points determined by certain collinearities. The definitions of three-space and of spaces of higher dimensions are direct generalizations of the definition of plane.† Since in the treatment of Veblen and Young the plane is defined as a class of points while point is an undefined symbol it is clear that the development cannot involve point and plane in precisely the same manner from the start. Indeed a considerable body of theorems must be proved before the general theorem of duality can be established. By a direct generalization of the theorems just mentioned and their proofs duality in spaces of higher dimensions is established.

The treatment of Veblen and Young has the obvious advantage that a small number of undefined symbols is used and that consequently the number of assumptions is small. On the other hand, it would seem desirable to treat point and plane as space duals in the assumptions themselves. While this requires a somewhat larger body of axioms it avoids some rather intricate and slippery argumentations at the very outset. This consideration will be of greater importance in case the assumptions are used as basis for a first course in projective geometry.

The purpose of this paper is to give an independent set of assumption which shall be sufficient for what Veblen and Young call general projective space.‡

The undefined elements are *point* and *plane*, each equally abstract and fundamental. Thus plane is not regarded as a class of points and the line does not occur explicitly at all in the assumptions. Point and plane are connected by two undefined relations "point *on* plane" and "plane *on* point." These are entirely independent except as noted on page 15. Thus "Point A is on plane α" need not mean "Plane α is on point A."

*A set of Assumptions for Projective Geometry, American Journal of Mathematics, vol. XXX (1908), pp. 347–380.

†See Veblen and Young, Projective Geometry, vol. I, pp. 29–33.

‡American Journal, vol. XXX, p. 347.

From the first equation a special value of u_0 is

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The use of this value of u_0 suggests that $(x - y)^{1-a}$ is a factor of the solution of the given equation. In like manner from the second equation

$$u_0 = \frac{-(x - y)^{1-b}}{1 - b},$$

which suggests that $(x - y)^{1-b}$ is a factor of the solution of the given equation.

From these suggestions it is inferred that $(x - y)^{1-a-b}$ is a factor of the solution of the given equation. This inference is correct, for the relation

$$u = (x - y)^{1-a-b} \cdot v$$

transforms the given equation into

$$\frac{\partial^2 v}{\partial x \partial y} - \frac{1 - b}{x - y}\frac{\partial v}{\partial x} + \frac{1 - a}{x - y}\frac{\partial v}{\partial y} = 0.$$

By the preceding paragraph a solution of this equation is

$$v = (x + m)^{a-1}(y + n)^{b-1}.$$

The corresponding solution of the given equation is

$$u = (x + m)^{a-1}(y + m)^{b-1}(x - y)^{1-a-b}.$$

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The treatment of Veblen and Young has the obvious advantage that a small number of undefined symbols is used and that consequently the number of assumptions is small. On the other hand, it would seem desirable to treat point and plane as space duals in the assumptions themselves. While this requires a somewhat larger body of axioms it avoids some rather intricate and slippery argumentations at the very outset. This consideration will be of greater importance in case the assumptions are used as basis for a first course in projective geometry.

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‡American Journal, vol. XXX, p. 347.

In general points are indicated by Roman Caps as A, and planes by small Greek letters as α.

1. A Set of Fundamental Propositions.

I_1. *The class of points contains at least three elements.*
I_2. *The class of planes contains at least three elements.*

II_1. *Three points are on at least one plane.*
II_2. *Three planes are on at least one point.*

III_1. *On three planes there is at least one point.*
III_2. *On three points there is at least one plane.*

IV_1. *If points A and B are on a plane α and C not on α, then A, B, C are on not more than one plane.*

IV_2. *If planes α and β are on a point A and γ not on A, then α, β, γ are on not more than one point.*

V_1. *Two points are on at least three planes.*
V_2. *Two planes are on at least three points.*

DEFINITIONS. Points which are on two planes are *collinear*.
Planes which are on two points are *collinear*.

VI_1. *Not all points on any one plane are collinear.*
VI_2. *Not all planes on any one point are collinear.*

Propositions $\mathrm{I}_1 \cdots \mathrm{VI}_2$ are clearly dual with respect to point and plane. That is, the propositions remain unchanged if the words *point* and *plane* are interchanged. Hence any proposition which is a formal consequence of these propositions remains valid when the words point and plane are interchanged.*

For proofs of the independence of this set of propositions see § 3.

2. Theorems.

The theorems of this section are all formal consequences of the assumptions of §1.

(1) THEOREM: *Not all points are on any one plane.*

Proof. Suppose all points on a plane α. By I_1 there are at least two points A and B on α and by V_1 A and B are on a plane β distinct from α. By VI_1 there is a point C on β and not on α. Hence not all points are an α.

(2) THEOREM: *If points A, B, C are on each of the distinct planes α and β and if A and B are on a third plane γ, then C is on γ.*

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*For a discussion of formal deduction from assumptions stated in terms of abstract symbols see E. V. Huntington, The Fundamental laws of Addition and Multiplication in Elementary Algebra, Annals of Math., Second series, vol. 8 (1906), pp. 1–49. Especially pp. 2–4.

Proof. If C is not on γ then the points A, B, C are on only one plane (IV_1) contrary to the hypothesis that A, B, C are on each of the distinct planes α and β.

(3) THEOREM: *If the points A, B, C, D are on the plane α and if no three of them are collinear then there is one and only one point E on α such that A, B, E and also C, D, E are collinear.*

Proof: By (1) a point F exists which is not on α and by II_1, IV_1 the points A, B, F are on one and only oneplane β and C, D, F are on one and only one plane γ.

By III_1 there is at least one point E on the planes α, β, γ.

If there is another point E' on α, β, γ then by IV_1 E, E', F are on only one plane. That is β and γ are the same plane and A, B, C, D are collinear.

(4) THEOREM: *If the points A, B, E are collinear and also C, D, E are collinear then A, B, C, D are on the same plane.*

Proof: Let A, B, C be on the plane α (II_1) and let F be a point not on α (1). Let B, E, F be on a plane β and D, E, F on a plane γ. Since A, B, E are collinear they must be on both α and β (2). Similarly C, D, E are on both α and γ. Hence A, B, C, D are all on α.

(5) THEOREM: (*a*) *Any two points are collinear.* (*b*) *If points A, B, C and also A, B, D are collinear then A, B, C, D are collinear.*

Proof: (*a*) is a direct consequence of V_1. (*b*) Let A, B, C be on two planes α and β, and let A, B, D be on planes α' and β'. Then by (2) C is on α' and also on β'. Hence A, B, C, D are collinear.

It will be noted that the above theorems are all stated in terms of "point on plane," and that in the proofs only assumptions with subscripts 1 are used. The duals of these theorems are:

1′. *Not all planes are on any one point.*

2′. *If planes α, β, γ are on each of the distinct points A and B and if α and β are on a third point C then γ is on C.*

3′. *If the planes α, β, γ, δ are on the point A and if no three of them are collinear then there is one and only one plane ϵ on A such that α, β, ϵ and also γ, δ, ϵ are collinear.*

4′. *If the planes α, β, ϵ are collinear and also γ, δ, ϵ are collinear then α, β, γ, δ are on the same point.*

5′. (*a*) *Any two planes are collinear.* (*b*) *If planes α, β, γ and also α, β, δ are collinear then α, β, γ, δ are collinear.*

3. Consistency of the Assumptions.

It is customary to inquire whether a given set of assumptions is consistent, independent and categorical.*

*For discussions of the properties of a set of assumptions denoted by these terms see Huntington, loc. cit.; Veblen, Trans. Am. Math. Society, vol. 5 (1904), pp. 343–384, especially p. 346, and references cited in these papers.

That a set of assumptions is consistent is proved by exhibiting a concrete system in which the assumptions are satisfied and which for some reason is regarded as self-consistent. The assumptions of this paper are satisfied by a set of fifteen planes and fifteen points with seven points on every plane and seven planes on every point. In this system "a plane is on a point" if "the point is on the plane" and conversely. In the following each column of seven letters represents a plane and each letter represents a point.

(A)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
A	A	A	B	B	C	C	D	D	E	E	F	F	G	G
B	B	B	C	C	D	D	E	E	F	F	G	G	A	A
C	H	M	K	N	I	H	O	M	L	K	J	H	I	K
D	D	D	E	E	F	F	G	G	A	A	B	B	C	C
E	I	L	L	H	K	J	H	K	I	O	O	L	J	H
F	J	N	M	I	L	M	J	I	N	H	K	I	O	L
G	K	O	J	O	O	N	L	N	J	M	N	M	M	N

To the set of assumptions given above further assumptions may be added in a variety of ways to give more special geometries. Thus the last four assumptions of the Veblen-Young set may be added to make it categorical as an ordinary projective geometry.

4. Independence of the Assumptions.

To prove that the propositions $I_1 \cdots VI_2$ of § 1 form an independent set of assumptions it is only necessary to prove I_1, II_1, III_1, IV_1, V_1, VI_1 independent, for it follows by the same arguments that $I_2, II_2, \cdots, VI_2$ are independent among themselves. This establishes the complete independence of the whole system, for the propositions with subscripts 1 cannot be used to prove a proposition with subscript 2 or vice versa since one set is stated in terms of "point on plane" and the other in terms of "plane on point." The propositions to be proved independent are:

I_1. The class of points contains at least three elements.

II_1. Three points are on at least one plane.

III_1. On three planes there is at least one point.

IV_1. If points P_1 and P_2 are on a plane α and P_3 not on α then P_1, P_2, P_3 are on not more than one plane.

V_1. Two points are on at least three planes.

VI_1. Not all points on any one plane are collinear.

These are shown to be independent by the following systems.

I_1. A system in which no point exists.

Propositions $II_1, \cdots, VI_1$ are to be regarded as hypothetical as regards the existence of points and planes. Thus II_1 written out in full would read "If points and planes exist three points are on at least one plane."

II_1. A system consisting of all planes in a projective geometry which are on one or more of seven points no four of which are coplanar and of all points on these planes.

III_1. A system consisting of the planes and points of Euclidian Geometry.

IV_1. A system consisting of system (A) given in §2, with the point K "on" the plane (3). Then ABK are on the distinct planes (2) and (3) but K is not on the plane (1) while A and B are on this plane.

V_1. A system in which the vertices of a tetrahedron $ABCD$ are points and any triad of these points are planes.

VI_1. A system consisting of three planes on each of which are the points A, B, C.

The assumptions of §1 together with the theorems deduced from them in §2 are sufficient to characterize a general projective space with the one exception that it does not follow from §1 that if a set of points is "on" a given plane then that plane is "on" those points. That is, "P_1 on α_1" need not mean "α_1 on P_1."

If now we define "P_1 on α_1" to mean the same as "α_1 on P_1" then the assumptions of §1 are far from independent. That they are still consistent follows from the finite system (A), p. 14. The redundancy of the assumptions in the presence of a reciprocal "on" was to be expected *a priori,* in as much as in that case the properties given in $\mathrm{I}_1 \cdots \mathrm{VI}_1$ in terms of "point on plane" certainly determine some properties in terms of "plane on point."

With a reciprocal "on" propositions I_1, I_2, II_1, II_2, IV_1, V_1, VI_1 form an independent set of assumptions while the remaining propositions are consequences of these. This we now proceed to prove.

III_1 and III_2 are immediate consequences of II_2 and II_1 respectively.

Proof of IV_2. Given a point A on planes α and β but not on γ. Suppose two points B and C on each of the planes α, β, γ. Since B and C are on γ but A not on γ, then A, B, C are on only one plane (IV_1), contrary to the assumption that A, B, C are on α and β.

Proof of V_2. Given any two planes α and β. By I_2, II_2 α and β are on a point A. Let D be a point on α but not on β (VI_1) and E a point on β but not on α. By V_1, three planes γ, δ, ϵ are on D and E. On α, β, γ there is a point A' (II_2); on α, β, δ a point B'; and on α, β, ϵ a point C'. Hence on α, β are the three points A', B', C'. If A', B', C' were not distinct at least two of the three planes γ, δ, ϵ would fail to be distinct.

Proof of VI_2. We need to show that on a given point A on which are two given planes α and β there is a plane γ not collinear with α and β. Let B be a point in α and not in β and C a point in β but not in α. Then the plane γ on A, B, C is not collinear with α and β.

To establish the independence of the remaining propositions of this set

we note that the systems given above to prove the independence of I_1, II_1, IV_1, V_1, VI_1 are sufficient to prove the independence of these propositions in the new set.

We next prove the independence of I_2, II_2.

I_2. A system containing no planes.

II_2. A system consisting of the points and planes of ordinary projective geometry with one point removed.

Columbia University, in the City of New York,
Jan. 28, 1911.

A FUNDAMENTAL PARAMETRIC REPRESENTATION OF SPACE CURVES.*

By Luther Pfahler Eisenhart.

Introduction.—In 1848 J. A. Serret † proposed the problem of solving the differential equation

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{1}$$

where s, x, y, z are functions of a single parameter, and he developed a general method of solution, without giving, however, in simple explicit form the general solution of this equation. The same problem has been considered by Darboux ‡ on two occasions; his solution makes use of the idea of curves with parallel tangents. Applying this method he gets rather complicated forms for the solution of (1). The author, while studying certain surfaces whose middle surfaces are surfaces of translation, was brought incidentally to the following form of solution

$$\begin{aligned} x &= \varphi - u\varphi' + \psi', \\ iy &= \varphi - u\varphi' - \psi', \\ z &= \varphi' + u\psi' - \psi, \end{aligned} \tag{2}$$

and

$$s = \varphi' - u\psi' + \psi, \tag{3}$$

where φ and ψ are functions of u and the primes indicate differentiation with respect to u. However, five years ago Montcheuil § was brought to a similar result in the same incidental manner. Salkowski ‖ made use of this result of Montcheuil in the discussion of certain problems. The present paper deals with an exposition of this form of defining a space

*Presented to the American Mathematical Society, April 28, 1911.

†Sur l'integration de l'equation $dx^2 + dy^2 + dz^2 = ds^2$, Journal de Mathématiques, vol. 13 (1848), pp. 353–360.

‡Sur l'integration de l'equation $dx^2 + dy^2 = ds^2$ et de quelques équations analogues, Journal de Mathématiques, ser. 2, vol. 18 (1873), pp. 236–240; also Sur la resolution de l'equation $dx^2 + dy^2 + dz^2 = ds^2$ et de quelques équations analogues, Journal de Mathématiques, ser. 4, vol. 3 (1887), pp. 305–325.

§Résolution de l'equation $ds^2 = dx^2 + dy^2 + dz^2$, Bulletin de la Société Mathématique de France, vol. 33 (1905), pp. 170–171.

‖Ueber algebraisch rektifizierbare Raumkurven, Mathematische Annalen, vol. 67 (1909), pp. 445–458.

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curve and with an investigation of certain problems for which this parametric form is very suitable. Before proceeding to an indication of the various topics discussed, it should be remarked that equations (2) and (3) give in explicit form the coördinates of minimal curves in four-space, which constitute an interesting generalization of the Weierstrass * formulas for minimal curves in three-space.

In §1 we show that it is possible to put the equations of any curve, *not a straight line*, in the form (2), and in general in two ways. We shall refer to (2) as the *normal form* of the equations, u the *normal parameter*, and φ and ψ the corresponding *normal functions*. Other normal parameters and functions of a similar type are discussed in §2. The expressions for the curvature, torsion and the direction-cosines of the tangent, principal normal and binormal are found in §3, together with a discussion of curves of zero curvature.

In §4 the general problem of the congruence of non-minimal curves is stated in terms of normal functions and parameters, and in §7 this problem is reduced to the integration of Schwarzian equations. The results are similar to those obtained recently by Study,† from a somewhat different point of view. In §8 the methods are applied to the study of the exceptional case of minimal curves. Curves in a real plane are considered in §5, and curves on the minimal cone in §6.

The question of algebraic curves is considered in §9, as well as the determination of a transformed form of equations (2) which is suitable for the discussion of real curves.

1. **Determination of** u, φ, ψ.—From (2) and (3) we have

$$\frac{dx + idy}{dz + ds} = -\frac{dz - ds}{dx - idy} = -u. \tag{4}$$

Suppose now that we have a curve defined by

$$x = f_1(v), \quad y = f_2(v), \quad z = f_3(v), \tag{5}$$

where f_1, f_2, f_3 are analytic functions of v for a certain domain of the latter. For the present we assume that

$$f_1 - if_2 \neq \text{const.} \tag{6}$$

If we put

$$t = (f_1'^2 + f_2'^2 + f_3'^2)^{\frac{1}{2}}, \tag{7}$$

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*Monatsberichte der Berliner Akademie (1866), p. 619; also Eisenhart, Differential Geometry, p. 49 (hereafter a reference to this book will be in the form E. p. 49).

†Zur Differentialgeometrie der analytischen Curven, Transactions of the American Mathematical Society, vol. 10 (1909), pp. 1–49; also Die Natürlichen Gleichungen der analytischen Curven in Euklidischen Raume, Transactions of the American Mathematical Society, voi. 11 (1910), pp. 249–279.

where the primes denote differentiation with respect to v, then in comparison with (4) we note that the functions u, φ, ψ defined by

$$\text{(8)} \qquad \frac{f'_3 - t}{f'_1 - if'_2} = u, \quad \tfrac{1}{2}(f_1 + if_2) = \varphi - u\frac{d\varphi}{du}, \quad \tfrac{1}{2}(f_1 - if_2) = \frac{d\psi}{du}.$$

enable us to write equations (5) in the form (2),* and thus are a normal parameter and normal functions.

Furthermore, the equations

$$\text{(9)} \qquad \frac{f'_3 + t}{f'_1 - if'_2} = \bar{u}, \quad \tfrac{1}{2}(f_1 + if_2) = \bar{\varphi} - \bar{u}\frac{d\bar{\varphi}}{d\bar{u}}, \quad \tfrac{1}{2}(f_1 - if_2) = \frac{d\bar{\psi}}{d\bar{u}},$$

determine a second normal parameter $\bar{u}$ and normal functions $\bar{\varphi}$ and $\bar{\psi}$ by means of which the equations of the curve can be given the normal form. It is evident from their manner of definition that there are only two sets of normal parameters and functions.

From (8) and (9) it follows that $u = \bar{u}$, when $t = 0$, that is when C is minimal. In this case,

$$\text{(10)} \qquad \varphi'' - u\psi'' = 0.$$

We proceed to the determination of the relations between the two sets u, φ, ψ and $\bar{u}$, $\bar{\varphi}$, $\bar{\psi}$. Equations (8) are identically satisfied by

$$\text{(11)} \qquad \begin{aligned} f_1 &= \varphi - u\varphi' + \psi', & if_2 &= \varphi - u\varphi' - \psi', \\ f_3 &= \varphi' + u\psi' - \psi, & t &= u\psi'' - \varphi'', \end{aligned}$$

and equations (9) reduce to

$$\text{(12)} \qquad \bar{u} = \frac{\varphi''}{\psi''}, \quad \bar{\varphi} - \bar{u}\bar{\varphi}' = \varphi - u\varphi', \quad \bar{\psi}' = \psi',$$

where the primes refer to differentiation with respect to the corresponding normal parameter. If the last two equations be differentiated with respect to u and the resulting equations be divided, the result is reducible by (12) to

$$\text{(13)} \qquad u = \frac{\bar{\varphi}''}{\bar{\psi}''},$$

which is in keeping with the symmetry of the equations.

From the last of (12) we have, by means of the first of (12),

$$\text{(14)} \qquad \bar{\psi} = \int \psi' d\bar{u} = \int \psi'\left(\frac{\varphi''}{\psi''}\right)' du = \psi'\left(\frac{\varphi''}{\psi''}\right) - \varphi',$$

*It should be observed that φ is determined thus only to within the additive function cu, where c is an arbitrary constant; the effect of taking $c \neq 0$ is merely to increase z by c. A similar effect is produced by the undetermined additive constant in ψ.

where we have disregarded an additive constant. In similar manner we have from (13)

$$\bar{\varphi} = \varphi - u\varphi' + (u\psi' - \psi)\frac{\varphi''}{\psi''}. \tag{15}$$

Furthermore from these values we find

$$\bar{s} = \bar{\varphi}' - \bar{u}\bar{\psi}' + \bar{\psi} = -s. \tag{16}$$

It remains for us to consider the exceptional case (6) which thus far has been excluded from the discussion. In this case we replace equations (8) and (9) by the unique set

$$\frac{f_1' + if_2'}{2f_3'} = -u, \quad \psi = cu + d, \quad \frac{d\varphi}{du} = f_3 + d, \tag{17}$$

where c and d denote constants.

Unless $t = 0$, equations (8) and (9) may be written

$$\frac{\gamma - 1}{\alpha - i\beta} = u, \quad \frac{\gamma + 1}{\alpha - i\beta} = \bar{u}, \tag{18}$$

where α, β, γ denote the direction-cosines of the tangent.

From (18) it follows that in the case of a non-minimal straight line, that is a curve whose tangent has the same direction at every point and for which $ds \neq 0$, both u and $\bar{u}$ are constant, and conversely. The same is true, furthermore, when the straight line is minimal, as follows from the general equations of such a line, namely *

$$x = \frac{1 - a^2}{2}v, \quad y = i\frac{1 + a^2}{2}v, \quad z = av, \tag{19}$$

with the difference that in this case u and $\bar{u}$ are equal to one another. Hence it is impossible to give a straight line parametric representation of the form (1).

We proceed to the consideration of the case where one of the normal parameters is constant, say $\bar{u} = c$ Then from (12) we have

$$\varphi'' = c\psi''. \tag{20}$$

In this case equations (2) reduce to

$$\begin{aligned} x &= c\psi - \psi'(cu - 1) + b, \\ iy &= c\psi - \psi'(cu + 1) + b, \\ z &= (c + u)\psi' - \psi + a, \end{aligned} \tag{21}$$

where a and b are constants. It is readily shown that the curve defined by (21) lies in the isotropic plane

*E. p. 48.

$$(22) \qquad (1 - c^2)x + i(1 + c^2)y + 2cz = A,$$

where A is a determinate constant. Conversely, equation (22) defines the general isotropic plane, and the condition that the functions (2) satisfy this equation is reducible to

$$(23) \qquad (u - c)(\varphi'' - c\psi'') = 0.$$

Hence (20) is the necessary and sufficient condition that the curve lie in an isotropic plane. It should be remarked that (17) corresponds to a special case of (20).

Gathering together these results we have the theorem:

When an analytic curve is defined by equations

$$x = f_1(v), \quad y = f_2(v), \quad z = f_3(v),$$

for values of v within a domain R, the functions u, φ, ψ determined by (8) *lead to a representation of the curve in the normal form. In general there is a second set of normalizing functions defined by*

$$(24) \qquad \bar{u} = \frac{\varphi''}{\psi''}, \quad \bar{\varphi} = \varphi - u\varphi' + (u\psi' - \psi)\frac{\varphi''}{\psi''}, \quad \bar{\psi} = \frac{\psi'\varphi''}{\psi''} - \varphi'.$$

When the curve is minimal, the two sets of functions are the same and conversely. When the curve lies in an isotropic plane, one of the normal parameters is constant, and conversely. For a straight line, minimal or otherwise, the representation is impossible.

In illustration of the foregoing we observe that for the circular helix

$$x = a \cos v, \quad y = a \sin v, \quad z = bv,$$

the normal parameters and functions are of the form

$$u = \frac{ic}{a}e^{iv}, \quad c = b - \sqrt{a^2 + b^2},$$

$$\varphi(u) = \frac{1}{2}\frac{a^2u}{ic}\left(1 - \log\frac{iau}{c}\right),$$

$$\psi(u) = \frac{i}{2}c\left(1 + \log\frac{iau}{c}\right).$$

The parameter $\bar{u}$ is obtained by replacing c by $\bar{c}$, where

$$\bar{c} = b + \sqrt{a^2 + b^2}.$$

The functions $\bar{\varphi}$ and $\bar{\psi}$ are of the same form as the above with $\bar{c}$ and $\bar{u}$ in place of c and u respectively.

2. **Other Normal Parameters.**—We have observed that there are at most two normal parameters in terms of which the equations of a curve may be given the form (2). In stating this fact we mean the precise form (2), for, as we shall show, there are other normal forms analogous to (2). In fact, we shall prove that it is possible to express the coordinates in a similar form in terms of a parameter u_1, defined by

$$u_1 = \frac{au + b}{cu + d},$$

where a, b, c, d are constants, which without loss of generality may be chosen to satisfy the condition

$$ad - bc = 1.$$

To this end we observe that the equations

$$(au + b)\varphi'' du = u_1\varphi_1''(u_1)du_1,$$

$$(cu + d)\varphi'' du = \varphi_1''(u_1)du_1,$$

and like ones in ψ and ψ_1, are consistent and may be replaced by

$$u_1 = \frac{au + b}{cu + d}, \quad \varphi_1(u_1) = \frac{\varphi}{cu + d}, \quad \psi_1(u_1) = \frac{\psi}{cu + d}.$$

Solving these equations for u, φ and ψ, we obtain

$$u = \frac{b - du_1}{cu_1 - a}, \quad \varphi = \frac{\varphi_1}{a - cu_1}, \quad \psi = \frac{\psi_1}{a - cu_1}.$$

When these values are substituted in (2) and (3), the latter become

$$x = d(\varphi_1 - u_1\varphi_1') + b\varphi_1' + a\psi_1' + c(\psi_1 - u_1\psi_1'),$$

$$iy = d(\varphi_1 - u_1\varphi_1') + b\varphi_1' - a\psi_1' - c(\psi_1 - u_1\psi_1'),$$

$$z = a\varphi_1' + c(\varphi_1 - u_1\varphi_1') - b\psi_1' - d(\psi_1 - u_1\psi_1'),$$

$$s = a\varphi_1' + c(\varphi_1 - u_1\varphi_1') + b\psi_1' + d(\psi_1 - u_1\psi_1'),$$

which expressions are a generalization of (2) and (3).

In order to interpret this result, we observe that if s be replaced by it in (3) and in the foregoing equations, these equations and (2) define in terms of the respective parameters u and u_1 a minimal curve in Euclidean four-space, as follows from (1).

If we denote by x_1, y_1, z_1, t_1 the coördinates of a minimal curve for

which u_1 is the normal parameter in the sense of §1 and φ_1, ψ_1 are the normal functions, so that

$$x_1 = \varphi_1 - u_1\varphi_1' + \psi_1', \quad iy_1 = \varphi_1 - u_1\varphi_1' - \psi_1',$$
$$z_1 = \varphi_1' + u_1\psi_1' - \psi_1, \quad it_1 = \varphi_1' - u_1\psi_1' + \psi_1,$$

the preceding set of equations is expressible in the form

$$\begin{aligned} x &= \tfrac{1}{2}[(a+d)x_1 + (d-a)iy_1 + (b-c)z_1 + (b+c)it_1], \\ y &= \tfrac{1}{2}[(a-d)ix_1 + (a+d)y_1 - (b+c)iz_1 + (b-c)t_1], \\ z &= \tfrac{1}{2}[(c-b)x_1 + (b+c)iy_1 + (a+d)\, z_1 + (a-d)it_1], \\ t &= \tfrac{1}{2}[-(b+c)ix_1 + (c-b)y_1 + (d-a)iz_1 + (a+d)t_1]. \end{aligned}$$

Since these equations define an orthogonal transformation, of determinant $+1$, of four-space, we have the theorem:

When the equations of a curve of Euclidean three-space are expressed in the form (2) in terms of a normal parameter u, it is possible without quadratures to express the coördinates in analogous form in terms of a parameter u_1, where

$$u_1 = \frac{au+b}{cu+d},$$

*a, b, c, d, being constants satisfying the condition $ad - bc = 1$. This transformation of parameters may be effected by the rotation of the minimal curve of coördinates $(x, y, z, -is)$ in Euclidean four-space.**

When the curve is defined in terms of a general parameter v as in §1, the parameter u_1 is given by

$$u_1 = \frac{a(f_3' - t) + b(f_1' - if_2')}{c(f_3' - t) + d(f_1' - if_2')}.$$

Moreover, the second parameter $\bar{u}^1$ is given by

$$\bar{u}_1 = \frac{a\bar{u}+b}{c\bar{u}+d} = \frac{a\varphi'' + b\psi''}{c\varphi'' + d\psi''} = \frac{a\varphi_1'' + b\psi_1''}{c\varphi_1'' + d\psi_1''}.$$

3. **Radii of Curvature and Torsion. Curves of Zero Curvature.**—The equation

$$\frac{1}{\rho^2} = \frac{\left(\frac{d^2x}{du^2}\right)^2 + \left(\frac{d^2y}{du^2}\right)^2 + \left(\frac{d^2z}{du^2}\right)^2 - \left(\frac{d^2s}{du^2}\right)^2}{\left(\frac{ds}{du}\right)^4}, \tag{25}$$

*The converse of this result is true, as will be shown in a subsequent paper dealing with minimal curves and surfaces in Euclidean four-space.

giving the radius of curvature of a curve, is independent of the real or complex character of the parameter u.* When the expressions from (2) and (3) are substituted in (25), it becomes

$$\frac{1}{\rho^2} = \frac{4(\varphi''' \psi'' - \varphi'' \psi''')}{(\varphi'' - u\psi'')^4}. \tag{26}$$

It is equally true that the derivation of the Frenet formulas †

$$\frac{d\alpha}{ds} = \frac{l}{\rho}, \quad \frac{dl}{ds} = -\left(\frac{\alpha}{\rho} + \frac{\lambda}{\tau}\right), \quad \frac{d\lambda}{ds} = \frac{l}{\tau} \tag{27}$$

is not conditioned by the character of the parameter u. In (27) α, l, λ denote the direction-cosines of the tangent, principal-normal and binormal with respect to the x-axis; with respect to the other two axes they are β, m, μ and γ, n, ν, respectively, and these direction-cosines also satisfy (27). Furthermore τ denotes the radius of torsion. Its expression, derived from‡

$$\frac{1}{\tau} = -\frac{\rho^2}{\left(\dfrac{ds}{du}\right)^6} \begin{vmatrix} \dfrac{dx}{du} & \dfrac{dy}{du} & \dfrac{dz}{du} \\ \dfrac{d^2x}{du^2} & \dfrac{d^2y}{du^2} & \dfrac{d^2z}{du^2} \\ \dfrac{d^3x}{du^3} & \dfrac{d^3y}{du^3} & \dfrac{d^3z}{du^3} \end{vmatrix}, \tag{28}$$

is reducible in terms of the normal functions and parameter to

$$\frac{1}{\tau} = i\left[\frac{d \log \rho}{ds} + \frac{(u\psi'' - \varphi'')'}{(u\psi'' - \varphi'')^2} + \frac{2\psi''}{(u\psi'' - \varphi'')^2}\right], \tag{29}$$

where the primes indicate differentiation with respect to u.

From (2) and (3) we find the following expressions for the direction-cosines of the tangent,

$$\alpha = (\psi'' - u\varphi'')A, \quad \beta = i(\psi'' + u\varphi'')A, \quad \gamma = (\varphi'' + u\psi'')A, \tag{30}$$

where for the sake of brevity we have put

$$A = (\varphi'' - u\,\psi'')^{-1}. \tag{31}$$

By means of (27) we obtain also

$$\begin{gathered} l = \rho A^3[B(1-u^2) - (\varphi''^2 - \psi''^2)], \quad m = i\rho A^3[B(1+u^2) + \varphi''^2 + \psi''^2], \\ n = 2\rho A^3[Bu + \varphi''\,\psi''], \end{gathered} \tag{32}$$

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*Cf. E. pp. 9, 10.
†E. p. 17.
‡E. p. 21, Ex. 11.

and

$$\lambda = -i\rho A^3[B(1-u^2)+(\varphi''^2-\psi''^2)], \quad \mu = \rho A^3[B(1+u^2)-(\varphi''^2+\psi''^2)], \tag{33}$$
$$\nu = -2i\rho A^3(Bu - \varphi''\psi''],$$

where we have put

$$B = \varphi''\psi''' - \varphi'''\psi''. \tag{34}$$

From (26) it follows that the necessary and sufficient condition that the first curvature of a curve be zero is that (20) be satisfied, or that either φ'' or ψ'' be zero. We have seen that when (20) is satisfied the curve lies in the plane (22). Furthermore, when φ'' or ψ'' is zero the curve lies in the plane $x + iy =$ const. or $x - iy =$ const. respectively. The converse of these results being true we have the theorem*

The necessary and sufficient condition that the first curvature of a curve be zero at all its points is that it be a straight line or a curve in an isotropic plane.

4. **Congruence of Space Curves.**—By analytical processes, which are independent of the character of the parameter in terms of which a given curve is defined, it can be shown that a necessary and sufficient condition that two non-minimal space curves be congruent, in the Euclidean sense, is that one of the following three sets of equations be satisfied by the linear element and the radii of curvature and torsion of the two curves:

$$\frac{d\rho}{ds} = f(\rho), \quad \tau = \varphi(\rho); \quad \rho = \text{const.}, \quad \frac{d\tau}{ds} = \varphi(\tau); \quad \rho = \text{const.}, \quad \tau = \text{const.},$$

the functions f and φ, or the constants as the case may be, being the same for both curves.†

If we exclude the case where either ρ or τ is constant, or both, it follows that if two curves are defined in terms of parameters v_1 and v_2 respectively the necessary and sufficient condition that the two curves be congruent, in the Euclidian sense, is that the equations

$$\rho_1(v_1) = \rho_2(v_2), \quad \tau_1(v_1) = \tau_2(v_2), \quad s_1(v_1) = s_2(v_2), \quad \frac{d\rho_1}{ds_1} = \frac{d\rho_2}{ds_2}$$

be consistent. Evidently one of the last two is a consequence of the other and of the first two. Hence, ρ, τ, s and $d\rho/ds$ constitute a set of *characteristic invariants,* to use the terminology of Study.‡

From (16) it is seen that the interchange of the normal parameters u and $\bar{u}$ effects a change of sign in the linear element of the curve. Hence

*Cf. E. Chapter 1.

†Lie, Vorlesungen uber Continuierliche Gruppen, Leipzig, 1893, pp. 684–686; also, Scheffers, Anwendung der Differential und Integral Rechnung auf Geometrie, Leipzig, 1902, vol. 1, p. 207.

‡L. c., p. 24.

the question of sign does not in general cause any difficulty in determining the curves congruent to a given curve, when the latter is defined by equations in the normal form. In consequence of (3), (26) and (29) we have accordingly the theorem:

The problem of finding curves congruent to a non-minimal space curve C, with its equations in the normal form, reduces to the solution of the equations obtained by equating the functions

$$\frac{\varphi'''\psi'' - \varphi''\psi'''}{(\varphi'' - u\psi'')^4}, \quad \frac{(u\psi'' - \varphi'')' + 2\psi''}{(u\psi'' - \varphi'')^2},$$

$$(35) \qquad \varphi' - u\psi' + \psi, \quad \left(\frac{\varphi'''\psi'' - \varphi''\psi'''}{(\varphi'' - u\psi'')^4}\right)'(\varphi'' - u\psi'')^{-1},$$

to similar expressions in u_1, φ_1, ψ_1, *where the latter are to be determined; of these equations the first two and either one of the last two constitute a sufficient set.* Incidentally we remark that the second of (35) is the function $i/\tau + 1/\rho\, d\rho/ds$, to within algebraic sign. Hence from the present point of view this invariant takes the place of τ which is usually used. A fuller significance of this will be seen later (§7).

5. **Curves in a Real Plane.**—If we put

$$\varphi' + u\psi' - \psi = 0,$$

the curve (2) lies in the plane $z = 0$. In this case equations (2) reduce to

$$x = 2\int\psi du - 2u\psi + (1 + u^2)\psi',$$

$$iy = 2\int\psi du - 2u\psi - (1 - u^2)\psi',$$

and the arc is given by

$$s = 2(\psi - u\psi').*$$

Furthermore, the curvature is given by

$$\rho^2 = -4u^4\psi''^2,$$

and the arc of the circular indicatrix of the tangent, namely $\sigma = \int\rho^{-1}ds$, is such that

$$u = e^{i\sigma}.$$

From the first equation of this section we find by differentiation that

$$\bar{u} = -u.$$

If the additive constant arising from the integration of the last of equations (12) be taken equal to zero, we have

$$\bar{\psi} = -\psi, \quad \bar{\psi} - \bar{u}\bar{\psi}' = -(\psi - u\psi'), \quad \bar{u}^2\bar{\psi}'' = -u^2\psi'',$$

*When the reader compares the expressions for x, y, z, with (67), he will observe that plane curves may be looked upon readily as minimal curves in three-space.

where in each case the primes indicate differentiation with respect to the argument.

From the foregoing results it follows that the determination of the curves congruent to a plane curve whose equations have the above form consists of the solution of the equations

$$\psi_1 - u_1\psi_1' = \psi - u\psi', \quad u_1^2\psi_1'' = u^2\psi'',$$

and of

$$\psi_1 - u_1\psi_1' = \psi - u\psi', \quad u_1^2\psi_1'' = -u^2\psi'',$$

where ψ_1 is a function of u_1.

The general solution of the former is

$$u_1 = cu, \quad \psi_1 = \psi + eu,$$

and of the latter

$$u_1 = \frac{c}{u}, \quad \psi_1 = \frac{2}{u}\int\psi\,du - \psi + \frac{e}{u},$$

where c and e are arbitrary constants.

When these values are substituted in equations for x_1 and y_1 analogous to those for x and y, the resulting equations are expressible in the forms

$$x_1 = \frac{1}{2c}[(1 + c^2)x - (1 - c^2)iy + 2e],$$

$$y_1 = \frac{1}{2c}[(1 - c^2)ix + (1 + c^2)y + 2ei ,$$

and

$$x_1 = \frac{1}{2c}[(1 + c^2)x + (1 - c^2)iy + 2e],$$

$$y_1 = \frac{1}{2c}[(1 - c^2)ix - (1 + c^2)y + 2ei],$$

respectively.

We shall close this section with the determination of plane curves of constant curvature $1/a$. From the expression for ρ it follows that such curves are characterized by the equation

$$4u^4\psi''^2 = -a^2,$$

from which we obtain

$$\psi = \frac{ai}{2}\log u + bu + c,$$

where b and c are constants.

Substituting in (2) we have

$$x = b + e + \frac{ai}{2u}(1 - u^2), \quad iy = e - b - \frac{ai}{2u}(1 + u^2),$$

so that

$$[x - (b + e)]^2 + [y - i(b - e)]^2 = a^2.$$

§6. **Curves on the Minimal Cone, or Sphere of Zero Radius.** Before considering further the problem of congruence of curves in general it is necessary to study in particular the curves which lie on the minimal cone, or sphere of zero radius, whose equation is

$$x^2 + y^2 + z^2 = 0. \tag{36}$$

The necessary and sufficient condition that a curve whose equations are in the normal form (2) lie on the cone (36) is

$$(\varphi' - u\psi' - \psi)^2 + 4\psi'(\varphi - u\psi) = 0. \tag{37}$$

If we put

$$f^2 = u\psi - \varphi, \tag{38}$$

the preceding equation reduces to

$$f^2(\psi' - f'^2) = 0. \tag{39}$$

When $f = 0$, and thus $\varphi = u\psi$, equations (2) become

$$x = (1 - u^2)\psi', \quad y = i(1 + u^2)\psi', \quad z = 2u\psi'. \tag{40}$$

Furthermore, from (26) we have

$$\frac{1}{\rho^2} = 4\frac{3\psi''^2 - 2\psi'\psi'''}{(2\psi')^4}, \tag{41}$$

and from (3)

$$s = 2\psi. \tag{42}$$

When $f \neq 0$ in (38), the corresponding equations (2) are reducible by means of (39) to

$$\begin{aligned} x &= -(uf' - f)^2 + f'^2, \\ iy &= -(uf' - f)^2 - f'^2, \\ z &= 2f'(uf' - f). \end{aligned} \tag{43}$$

When one applies (24) to these equations and determines the functions $\bar{u}$, $\bar{\varphi}$, and $\bar{\psi}$, he obtains

$$\bar{u} = u - \frac{f}{f'}, \quad \bar{\varphi} = \frac{1}{f'}(ff' - \psi)(uf' - f), \quad \bar{\psi} = ff' - \psi.$$

From these follows the relation $\bar{\varphi} - \bar{u}\bar{\psi} = 0$, and consequently the expressions (43) are transformed into the forms (40). Conversely, it is readily shown that in terms of the second set of parameters and functions the equations (40) assume the form (43). Hence we have the theorem:

The curves on the cone $x^2 + y^2 + z^2 = 0$ are characterized by equations of the form (40) and there is only one such representation of the curve for a given set of coördinate axes.

We pass to the consideration of the congruence of two curves defined by equations of the form (40). In this case the expressions (35) reduce respectively to

$$\text{(44)} \qquad \frac{3\psi''^2 - 2\psi'\psi'''}{(2\psi')^4}, \quad 0, \quad 2\psi, \quad \left(\frac{3\psi''^2 - 2\psi'\psi'''}{(2\psi')^4}\right)' \frac{1}{2\psi'}.$$

If equation (42) be differentiated with respect to s, we obtain

$$\text{(45)} \qquad 2\frac{d\psi}{du}\frac{du}{ds} = 1,$$

in consequence of which we find that

$$\frac{3\psi''^2 - 2\psi'\psi'''}{(2\psi')^4} = \frac{2u'u''' - 3u''^2}{4u'^2},$$

where the primes denote differentiation with respect to u and s respectively in the left-hand and right-hand members of the equation. Consequently in this case the general theorem of §4 assumes the following form similar to a theorem due to Study:*

To each Schwarzian equation

$$\text{(46)} \qquad \{u, s\} = \frac{2u'u''' - 3u''^2}{2u'^2} = \tfrac{1}{2}f(s),$$

determined by a function $f(s)$, there belongs a class of congruent curves, lying on the cone $x^2 + y^2 + z^2 = 0$, each curve being determined by a solution u of this equation; the function $f(s)$ is the square of the first curvature of the curve; and the equations of the curve are

$$\text{(47)} \qquad x = \frac{1 - u^2}{2u'}, \quad y = i\frac{(1 + u^2)}{2u'}, \quad z = \frac{u}{u'}.$$

It is readily found that these coordinates are solutions of the equation

$$\text{(48)} \qquad \theta''' + \varphi\theta' + \tfrac{1}{2}\varphi'\theta = 0,$$

where the primes indicate differentiation with respect to s.

There is a result analogous to the preceding arising from the consideration of equations (43). Corresponding to (41) and (42) we have

$$\text{(49)} \qquad \rho^2 = f^3 f'', \quad s = 2(\psi - ff').$$

Since the second of the expressions (35) vanishes identically, the foregoing constitute a set of characteristic invariants. From (49) we have

$$\frac{ds}{\rho^2} = -2\frac{du}{f^2},$$

*L. c., p. 253.

so that if we introduce a parameter σ defined by

$$\sigma = \tfrac{1}{2}\int \frac{ds}{\rho^2}, \tag{50}$$

and indicate by primes derivatives with respect to σ, we have

$$\{u, \sigma\} = \frac{2u'u''' - 3u''^2}{2u'^2} = 2\rho^2, \tag{51}$$

and equations (43) become

$$\begin{aligned} x &= -\left(1 - \frac{uu''}{2u'^2}\right)^2 u' + \frac{u''^2}{4u'^3}, \\ y &= i\left(1 - \frac{uu''}{2u'^2}\right)^2 u' + \frac{iu''^2}{4u'^3}, \\ z &= \frac{u''}{u'}\left(\frac{uu''}{2u'^2} - 1\right). \end{aligned} \tag{52}$$

Combining these results with the preceding theorem, we obtain the result:

To each solution of a Schwarzian equation

$$\{u, t\} = \tfrac{1}{2}f(t),$$

there correspond two curves on the cone $x^2 + y^2 + z^2 = 0$: *for one of them* t *is the arc and* $f(t)$ *the square of the first curvature; for the other* t *is equal to*

$$\frac{1}{2}\int \frac{ds}{\rho^2}$$

and $f(t)$ *is four times the square of the radius of curvature; when a solution is known the finite equations of the curve can be found without quadrature.*

7. Curves in General which are not Minimal.—The foregoing theory of curves on a minimal cone may be applied to any non-minimal curves, since there are associated with any such curve two simple curves on a minimal cone. The coördinates of these curves are

$$\xi = \rho(l + i\lambda), \quad \eta = \rho(m + i\mu), \quad \zeta = \rho(n + i\nu), \tag{53}$$

$$\bar{\xi} = \rho(l - i\lambda), \quad \bar{\eta} = \rho(m - i\mu), \quad \bar{\zeta} = \rho(n - i\nu). \tag{54}$$

Evidently the factor ρ is unessential as far as determining the fact that these curves lie on the cone $x^2 + y^2 + z^2 = 0$. However, this factor enables one to put the expressions in the desired form, when the values for

l, m, n; λ, μ, ν from (32), (33) are substituted. In fact, they reduce by means of (12) to

$$\xi = (1 - u^2)\psi_1', \quad \eta = i(1 + u^2)\psi_1', \quad \zeta = 2u'\psi_1, \tag{55}$$

$$\bar{\xi} = (1 - \bar{u}^2)\bar{\psi}_1', \quad \bar{\eta} = i(1 + \bar{u}^2)\bar{\psi}_1', \quad \bar{\zeta} = 2\bar{u}\bar{\psi}_1', \tag{56}$$

where

$$\psi_1' = \tfrac{1}{2}(u\psi'' - \varphi''), \quad \bar{\psi}_1' = \tfrac{1}{2}(\bar{u}\bar{\psi}'' - \bar{\varphi}''). \tag{57}$$

From (42) it follows that the arcs s_1 and $\bar{s}_1$ of the curves (55), (56) are given by

$$s_1 = 2\psi_1 = -(s + c), \quad \bar{s}_1 = 2\bar{\psi}_1 = -(\bar{s} + \bar{c}). \tag{58}$$

Since these additive constants c, $\bar{c}$ may be equated to zero, we have the following theorem:

Any analytic curve, which is not minimal, may be represented upon the minimal cone $x^2 + y^2 + z^2 = 0$, in two ways, such that the curvilinear distance of two points on each conical curve is equal to the curvilinear distance between the corresponding points on the given curve. We shall speak of the curves (55), (56) as the *conical representations* of the given curve.

When the expression (57) for ψ_1' is substituted in an equation similar to (41), we obtain the following expression for the curvature of (55)

$$\frac{1}{\rho_1^2} = \frac{1}{\rho^2} - 2\frac{dM}{ds} - M^2, \tag{59}$$

where M denotes the differential invariant

$$M = \frac{i}{\tau} + \frac{1}{\rho}\frac{d\rho}{ds}. \tag{60}$$

From (59) and (46), we have that the determination of curves on the minimal cone which are congruent to the conical representation (55) of a given curve requires the integration of the Schwarzian equation

$$\{u, s\} = \frac{2u'u''' - 3u''^2}{2u'^2} = \frac{1}{2}\left(\frac{1}{\rho^2} - 2\frac{dM}{ds} - M^2\right), \tag{61}$$

where s denotes the arc of the given curve.

In consequence of (57) and (45), the first of equations (58) may be written

$$\varphi'' - u\psi'' = \frac{1}{u'}, \tag{62}$$

where the primes denote differentiation with respect to u and s respect-

ively. Differentiating this equation with respect to s, we obtain

$$(63)\qquad (\varphi'' - u\psi'')' = -\frac{u''}{u'^3},$$

and consequently equation (29) may be written

$$(64)\qquad \psi'' = -\frac{1}{2u'^2}\left(M + \frac{u''}{u'}\right),$$

where primes have the same significance as in (62) and (63).

Conversely, suppose that we have a solution u of equation (61). From (62) and (64) we obtain two functions φ and ψ, which when substituted in (26) and (29) show that the curve determined by these functions has the same intrinsic equations as the given curve. Hence we have the theorem:*

The determination of all curves congruent to a given curve requires the integration of the Schwarzian equation (61) *and quadratures.*

For the other conical representation we have analogous to (61) the equation

$$(65)\qquad \{\bar{u}, s\} = \frac{2\bar{u}'\bar{u}''' - 3\bar{u}''^2}{2\bar{u}'^2} = \frac{1}{2}\left(\frac{1}{\rho^2} + 2\frac{d\bar{M}}{ds} - \bar{M}^2\right),$$

where

$$\bar{M} = \frac{1}{\rho}\frac{d\rho}{d\bar{s}} + \frac{i}{\tau} = -\frac{1}{\rho}\frac{d\rho}{ds} + \frac{i}{\tau}.$$

From (62) and (64) it follows that the solution of (65) corresponding to a solution u of (61) is given by

$$(66)\qquad \bar{u} = u - 2u'^2(Mu' + u'')^{-1}.$$

8. **Congruence of Minimal Curves.** By means of the theory of curves on the minimal cone we are able also to handle the exceptions to the theorems of §4 and to establish a criterion for the congruence of minimal curves.

If we take the equations of a minimal curve in the Weierstrass form †

$$(67)\qquad \begin{aligned} x &= (1 - u^2)f'' + 2uf' - 2f, \\ iy &= -(1 + u^2)f'' + 2uf' - 2f, \\ z &= 2(uf'' - f'), \end{aligned}$$

we obtain on differentiation

$$(68)\qquad \frac{dx}{du} = (1 - u^2)f''', \quad i\frac{dy}{du} = -(1 + u^2)f''', \quad \frac{dz}{du} = 2uf'''.$$

*Cf. Study, l. c., pp. 259, 260.

†L. c.

If we put

$$d\sigma = \sqrt{2f'''}du, \tag{69}$$

it is readily found that

$$\left(\frac{d^2x}{d\sigma^2}\right)^2 + \left(\frac{d^2y}{d\sigma^2}\right)^2 + \left(\frac{d^2z}{d\sigma^2}\right)^2 = 1.$$

Hence σ is the arc of the following curve

$$X = \frac{1-u^2}{\sqrt{2}}\sqrt{f'''}, \quad Y = \frac{i(1+u^2)}{\sqrt{2}}\sqrt{f'''}, \quad Z = \frac{2u}{\sqrt{2}}\sqrt{f'''}, \tag{70}$$

which evidently lies on the cone $x^2 + y^2 + z^2 = 0$. In § 6 it was seen that a given curve on this cone admits of only one such representation for given rectangular axes. In like manner it may be shown that a given minimal curve admits of only one such set of equations as (67) for given coördinate axes. Hence the problems of the congruence of minimal curves and of the curves (70) are equivalent.

Comparing equations (70) with (40) we have

$$\psi' = \frac{\sqrt{f'''}}{\sqrt{2}},$$

from which it follows that

$$\frac{2\psi'\psi''' - 3\psi''^2}{(2\psi')^4} = \frac{4f'''f^{\text{V}} - 5f^{\text{IV}2}}{32f'''^3}.$$

$$\left(\frac{2\psi'\psi''' - 3\psi''^2}{(2\psi')^4}\right)'\frac{1}{2\psi'} = \frac{\sqrt{2}}{64(f''')^{\frac{9}{2}}}(4f'''^2f^{\text{VI}} - 18f'''f^{\text{IV}}f^{\text{V}} + 15f^{\text{IV}3}).$$

Hence from (44) we have the following theorem:*

The functions

$$J_5 = \frac{4f'''f^{\text{V}} - 5f^{\text{IV}2}}{4f'''^3},$$

$$J_6 = \frac{4f'''^2f^{\text{VI}} - 18f'''f^{\text{IV}}f^{\text{V}} + 15f^{\text{IV}3}}{(f''')^{\frac{9}{2}}}$$

constitute a set of characteristic invariants for a minimal curve.

9. **Algebraic Curves.—Real Curves.** It is evident that if φ and ψ in equation (2) are algebraic functions of u the curve is algebraic. We consider the converse problem.

Suppose that a curve is defined by two algebraic equations

$$f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0.$$

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*Lie, l. c., p. 704. Also Vessiot, Sur les courbes minima, Comptes rendus, vol. 140 (1905), pp. 1381–1384. Also Study, l. c., p. 39 and pp. 253–255.

Then y and z are expressible as algebraic functions of x, and consequently u as defined by an equation similar to the first of (8) is an algebraic function of x. In like manner y and z may be shown to be algebraic functions of u. Hence we have the theorem:

The necessary and sufficient condition that a curve defined by equations in the normal form be algebraic is that the coördinates be algebraic functions of the normal parameter.

An immediate consequence of this theorem is that the functions

$$\varphi - u\varphi', \quad \psi', \quad \varphi' - \psi, \qquad \varphi - u\psi$$

must be algebraic functions of u. To these may be added the function $s - 2\psi$, from which follows the theorem:

The necessary and sufficient condition that the arc of an algebraic curve be expressible algebraically in terms of the coördinates of its end points is that φ and ψ be algebraic functions of u.

Algebraic curves of this sort have been called *algebraically rectifiable* by Stäckel.*

Since φ'' and ψ'' are algebraic functions of u for all algebraic curves, and consequently ρ and τ are algebraic, we have the theorem of Salkowski:†

Every algebraic curve which is characterized by an algebraic intrinsic equation $f(s, \rho\ \tau,) = 0$ is algebraically rectifiable.‡

In certain cases it is desirable to know whether or not a curve is real. We shall show that there exists a transformed set of equations in which the functions and parameter are real.

From (2) and (3) it follows that the functions

$$\varphi', \quad u\psi' - \psi, \quad \varphi - u\varphi' + \psi', \quad i(\varphi - u\varphi' - \psi') \tag{71}$$

are real for a real curve, and conversely. If we put

$$u = p + iq,$$

then from (8) it would be possible to determine p and q as functions of the general parameter v, and thus we should obtain a relation between p and q. This is unessential except in that it tells us that we may put

$$\varphi(u) = P'(p) + iQ'(q),$$

*Ueber algebraisch rectifizierbare Raumcurven, Mathematische Annalen, vol. 43 (1893), pp. 171–184. The fundamental theorem of this paper is the following: The necessary and sufficient condition that an algebraic curve be algebraically rectifiable is that it be an evolute of an algebraic curve for which the sine of the total torsion, namely $\int \frac{ds}{\tau}$, is an algebraic function of the coördinates.

†L. c., p. 446.

‡The question of algebraically rectifyable curves has been touched upon also by Darboux, l. c., p. 316.

where the primes indicate differentiation with respect to the argument (the significance of this choice will be seen presently). Since φ' is real, it follows from the equation

$$d\varphi = \varphi'(dp + idq) = P''dp + iQ''dq,$$

that

$$\varphi' = P'' = Q'' = \sigma,$$

where σ is introduced for the sake of subsequent brevity. From these results and the fact that the last two of (71) must be real we find that

$$\psi' = P' - pP'' - i(Q' - qQ'')$$

and finally

$$\psi = 2P - pP' + 2Q - qQ' + i(qP' - pQ').$$

When these values are substituted in (2) and (3) we obtain

$$x = 2(P' - p\sigma), \qquad y = 2(Q' - q\sigma),$$

$$z = \sigma(1 - p^2 - q^2) + 2(pP' - P + qQ' - Q),$$

$$s = \sigma(1 + p^2 + q^2) - 2(pP' - P + qQ' - Q).*$$

In terms of these functions, we have

$$\varphi'' - u\psi'' = \frac{P'''Q'''(1 + p^2 + q^2)}{Q''' + iP'''},$$

$$\frac{1}{\rho^2} = \frac{4}{1 + p^2 + q^2}\left(\frac{1}{P'''^2} + \frac{1}{Q'''^2}\right).$$

$$\bar{u} = \frac{\varphi''}{\psi''} = -\frac{1}{p - iq}.$$

The last of these is an evident consequence of the definition of $\bar{u}$ and u.

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*Cf. Montcheuil, l. c.

GENERALIZATIONS IN THE THEORY OF NUMBERS AND THEORY OF LINEAR GROUPS.

By Mildred Sanderson.

1. Condition for an inverse. — The term function is here used to denote a rational integral function of y with integral coefficients. Employing a fixed integer m and a fixed function $P(y)$, we shall say that two functions are congruent modulis m and $P(y)$ if their difference can be given the form $mq(y) + P(y)Q(y)$; also that $f(y)$ has an inverse $f_1(y)$ if $f(y) \cdot f_1(y)$ is congruent to unity modulis m, $P(y)$. Then $f(y)$ and $f(y) + k(y)P(y)$ have the same inverse, so that we may restrict attention to functions of degree less than the degree r of $P(y)$. We proceed to prove the

Theorem. *If $P(y)$ is of degree r and is irreducible with respect to each prime factor of m, a function $R(y)$ of degree $< r$ has an inverse modulis m and $P(y)$ if and only if the greatest common divisor d of the coefficients of $R(y)$ is prime to m.*

We have $R(y) = dF(y)$. For any function $R_1(y)$, we may write

$$R(y)R_1(y) = dR_2(y) + P(y)Q(y),$$

where $R_2(y)$ is of degree $< r$. If R_1 is the inverse of R, then $dR_2(y) \equiv 1 \pmod{m}$, identically in y, so that d must be prime to m.

Conversely, if d is prime to m, $R(y)$ has an inverse modulis m, $P(y)$. We first prove by induction that $R(y)$ has an inverse modulis p^e, $P(y)$, where p is any prime factor of m, and e any positive integer. This is a well known fact for the case* $e = 1$. Assume that $R(y)$ has the inverse $R_1(y)$ modulis p^{e-1}, $P(y)$, so that

$$RR_1 = 1 + a(y)p^{e-1} + A(y)P(y).$$

Since R has an inverse modulis p, $P(y)$, we can choose $S(y)$ so that

$$RS(y) = -a(y) + pf(y) + F(y)P(y).$$

Then R is seen to have the inverse $R_1 + Sp^{e-1}$ modulis p^e, $P(y)$.

It remains to prove that if $R(y)$ has the inverse R_1 modulis m_1, $P(y)$, and the inverse R_2 modulis m_2, $P(y)$, where m_1 and m_2 are relatively prime,

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*Serret, Algèbre, vol. 2, ch. 3, sec. 3; Dickson, Linear Groups, § 7.

then $R(y)$ has an inverse modulis $m = m_1m_2, P(y)$. Set

$$RR_1 = 1 + m_1a_1(y) + A_1(y)P(y), \quad RR_2 = 1 + m_2a_2(y) + A_2(y)P(y).$$

Then

$$R(m_2R_1 - m_1R_2) = m_2 - m_1 + m(a_1 - a_2) + (m_2A_1 - m_1A_2)P(y).$$

Since $m_2 - m_1$ is prime to m, we can determine an integer k such that $k(m_2 - m_1) \equiv 1 \pmod{m}$. Then $k(m_2R_1 - m_1R_2)$ is an inverse of R modulis $m, P(y)$.

2. **Number of Classes of Residues Having an Inverse.**—Any function of y is congruent modulis $m, P(y)$ to a residue

$$a(y) = a_0 + a_1y + \cdots + a_{r-1}y^{r-1}$$

each of whose coefficients a_i belongs to the set $0, 1, \cdots, m - 1$. The number of ways of choosing r integers a_i from $0, 1, \cdots, m - 1$, such that the greatest common divisor of the a_i is prime to m, is*

$$(1) \qquad n = [m, r] \equiv m^r\left(1 - \frac{1}{p_1^r}\right)\left(1 - \frac{1}{p_2^r}\right)\cdots\left(1 - \frac{1}{p_s^r}\right),$$

where $p_1, \cdots, p_s$ are the distinct prime factors of m. Hence there are exactly n classes of residues moduli $m, P(y)$, each having an inverse. The notation $\phi_r(m)$ is often used for this important generalization $[m, r]$ of Euler's function $\phi(m)$.

3. **Generalization of Fermat's Theorem.**—*If the remainder of degree $< r$ obtained on dividing $f(y)$ by $P(y)$ has coefficients whose greatest common divisor is prime to m, and if $P(y)$ is irreducible with respect to each prime factor of m, then*

$$(2) \qquad f^{[m, r]} \equiv 1 \ [\textit{modd } m, P(y)].$$

Denote by $R_1, \cdots, R_n$ the distinct residues having inverses modulis $m, P(y)$. Then $R_1R_i, \cdots, R_nR_i$ are congruent to $R_1, \cdots, R_n$ in some order. Comparing the products, we get $R_i^n \equiv 1$ [modd $m, P(y)$].

For the case in which m is a prime p, we have $n = [p, r] = p^r - 1$. The theorem is thus a generalization also of Galois' theorem that

$$(3) \qquad f^{p^r-1} \equiv 1 \ [\text{modd } p, P(y)],$$

if $f(y)$ is not divisible by $P(y)$ modulo p, and $P(y)$ is irreducible modulo p.

4. **Two-fold Generalization of Wilson's Theorem.**—*The product of the distinct residues $R_1, \cdots, R_n$, having inverses modulis $m, P(y)$, is congruent to*

*C. Jordan, Traité des substitutions, § 124.

-1 *when m is a power of an odd prime or twice the power of an odd prime, or when $r = 1$, $m = 4$. In all other cases, the product is congruent to $+1$ modulis m, $P(y)$.*

The product is congruent to $(-1)^{s/2}$, where s is the number of residues R_i whose square is congruent to unity. The proof is analogous to that of Gauss' generalization to any composite integral modulus of Wilson's theorem.

5. **Theorem.** *Let $A(y)$ and $B(y)$ be functions each of degree less than the degree of $P(y)$, which is irreducible with respect to each prime factor of m. If m and the coefficients of $A(y)$, $B(y)$ do not all have a common factor, there exist functions $\alpha(y)$, $\beta(y)$ such that*

$$\alpha(y)A(y) + \beta(y)B(y) \equiv 1 \quad [\textit{modd}\ m, P(y)]. \tag{4}$$

Let $A(y) = aA_1(y)$, $B(y) = bB_1(y)$, where the greatest common divisor of the coefficients of $A_1(y)$ is prime to m, likewise that for $B_1(y)$. Since the greatest common divisor of a, b, m is 1, there exist integers a_1, b_1 for which $a_1a + b_1b \equiv 1 \pmod m$. Then $\alpha(y) = a_1A_1^{-1}(y)$, $\beta(y) = b_1B_1^{-1}(y)$ satisfy (4).

6. **Theorem.** *There exists a function $P(y)$ of any assigned degree r which is irreducible with respect to any assigned prime moduli $p_1, \cdots, p_s$.*

As well known, there exists a function $P_i(y)$ of degree r irreducible modulo p_i. We may take

$$P(y) = \sum_{i=1}^{s} p_1 \cdots p_{i-1}p_{i+1} \cdots p_sP_i(y). \tag{5}$$

7. **Generalized Linear Substitutions.**—We consider substitutions

$$x_i' \equiv \sum_{j=1}^{v} c_{ij}(y)x_j \ [\text{modd}\ m, P(y)] \qquad (i = 1, \cdots, v), \tag{6}$$

in which the $c_{ij}(y)$ are rational integral functions of y with integral coefficients such that the determinant $|c_{ij}(y)|$ has an inverse modulis m, $P(y)$. Then the substitution has an inverse. Every such substitution is the product of substitutions of two elementary types, the one altering only one variable x_i, replacing it by $x_i + c(y)x_j$; the other altering only one variable, multiplying it by a function $l(y)$ having an inverse.

The order of the group $G(m, r, v)$ of all the substitutions (6) is

$$\Omega(m, r, v) = [m, rv]m^{r(v-1)}[m, r(v-1)]m^{r(v-2)} \cdots [m, r].$$

If $m = m_1m_2$, where m_1 and m_2 are relatively prime, the group G is the direct product of the permutable groups H_1, H_2, where H_k is the group composed of the substitutions

$$x_i' \equiv x_i + m_k \Sigma\, d_{ij}(y)x_j \ [\text{modd}\ m, P(y)].$$

The group H_1 is simply isomorphic with the group $G(m_2, r, v)$.

It remains to treat the case in which $m = p^e$, where p is a prime. The factors of composition of $G(p^e, r, v)$ are those of $G(p, r, v)$ and a certain number of p's.

The proofs of the preceding results are similar to that for the case of a single modulus m, C. Jordan, Traité des substitutions, pp. 93–105. The final group $G(p, r, v)$ has been discussed by L. E. Dickson, Linear Groups, p. 81, and Annals of Mathematics, 1, vol. 11 (1897), p. 169.

THE UNIVERSITY OF CHICAGO, May, 1911.

TRANSFORMATION OF SERIES BY MEANS OF FUNCTIONS ADMITTING A RECURRENT RELATION.

By W. C. Brenke.

1. By transformation of trigonometric series certain results have been obtained as to the behavior of these series.* We wish to call attention to a generalization contained in the following theorem.

Theorem 1. *Let the function* ψ_n *satisfy a recurrent relation of the form*

$$\varphi_n \psi_n = \alpha_n \psi_{n+r} + \beta_n \psi_{n+s}, \qquad (n = 1, 2, 3, \cdots), \tag{1}$$

where all the expressions involved are defined for each value of n, *and* φ_n *does not vanish identically for any value of* n. *They may be constants or functions of a variable,* x. *The numbers* r *and* s, *which are independent of* n, *are not restricted to integral values, but* $r - s$ *is a positive integer.*

Then the series $\sum_1^\infty u_n \psi_n$, *where* u_n *is defined for each value of* n, *converges provided that*

(α) *the series* $\sum_{n=1}^\infty \left(\frac{\alpha_n}{\varphi_n} u_n + \frac{\beta_{n+r-s}}{\varphi_{n+r-s}} u_{n+r-s} \right) \psi_{n+r}$ *converges, and that*

(β) *either* $\lim_{n=\infty} \frac{\beta_n}{\varphi_n} u_n \psi_{n+s} = 0$, *or that* $\lim_{n=\infty} \frac{\alpha_n}{\varphi_n} u_n \psi_{n+r} = 0$.

We then have

$$\sum_1^\infty u_n \psi_n = \sum_{n=1}^{r-s} \frac{\beta_n}{\varphi_n} u_n \psi_{n+s} + \sum_{n=1}^\infty \left(\frac{\alpha_n}{\varphi_n} u_n + \frac{\beta_{n+r-s}}{\varphi_{n+r-s}} u_{n+r-s} \right) \psi_{n+r}. \tag{2}$$

To obtain this result we replace ψ_n in the polynomial $\sum_1^k u_n \psi_n$ by its value from (1), so that

$$\sum_{n=1}^k u_n \psi_n = \sum_{n=1}^k \left(\frac{\alpha_n}{\varphi_n} \psi_{n+r} + \frac{\beta_n}{\varphi_n} \psi_{n+s} \right) u_n = \sum_{n=1}^k \frac{\alpha_n}{\varphi_n} u_n \psi_{n+r} + \sum_{n=1-r+s}^{k-r+s} \frac{\beta_{n+r-s}}{\varphi_{n+r-s}} u_{n+r-s} \psi_{n+r};$$

hence,

$$\sum_{n=1}^k u_n \psi_n = \sum_{n=1}^{r-s} \frac{\beta_n}{\varphi_n} u_n \psi_{n+s} + \sum_{n=1}^k \left(\frac{\alpha_n}{\varphi_n} u_n + \frac{\beta_{n+r-s}}{\varphi_{n+r-s}} u_{n+r-s} \right) \psi_{n+r} - \sum_{n=k+1}^{k+r-s} \frac{\beta_n}{\varphi_n} u_n \psi_{n+s}; \tag{3}$$

if we allow k to increase indefinitely, the theorem follows.

* Schlömilch: Comp. d. höh. Analysis, vol. 1, § 40; Lerch: Annales de l'Ecole normale, ser. 3, vol. 12 (1895), p. 351; the writer: Annals of Mathematics, ser. 2, vol. 8 (1907), p. 87.

When the given series $\sum_1^\infty u_n\psi_n$ is known to converge, (β) is a necessary and sufficient condition that the transformed series (α) shall converge and to the value as given by (2).

DEFINITION. We shall refer to the results expressed in (2) as "a transformation of the series $\sum_1^\infty u_n\psi_n$ by the factor φ_n." If the series in the right-hand member of (2) be similarly treated and the process repeated p times, we define the final result as "a transformation of the original series by the factor φ_n^p."

APPLICATION. In practical applications equation (2) furnishes a convergence test for the series on the left, and also facilitates its approximate numerical evaluation, since it is often possible to transform a given series into another which converges more rapidly. Some series can be summed in closed form by this process.

We shall consider cases in which ψ_n is one of the functions $\cos nx$, the nth polynomial of Hermite* $U_n(x)$, the nth polynomial of Legendre $X_n(x)$, or the nth Bessel function $J_n(x)$. In place of (1) we take respectively,

$$(a)\qquad 2\cos\tfrac{1}{2}hx\cos nx = \cos(n+\tfrac{1}{2}h)x + \cos(n-\tfrac{1}{2}h)x;$$

$$(b)\qquad -U_n(x) = \frac{1}{2n+2}U_{n+2}(x) + \frac{2x}{2n+2}U_{n+1}(x);$$

$$(c)\qquad xX_n(x) = \frac{n+1}{2n+1}X_{n+1}(x) + \frac{n}{2n+1}X_{n-1}(x):$$

$$(d)\qquad \frac{2}{x}J_n(x) = \frac{1}{n}J_{n+1}(x) + \frac{1}{n}J_{n-1}(x).$$

Then from (2) we have

$$(a_1)\qquad \sum_1^\infty u_n\cos nx = \frac{1}{2\cos\frac{1}{2}hx}\sum_{n=1}^{h} u_n\cos(n-\tfrac{1}{2}h)x + \frac{1}{2\cos\frac{1}{2}hx}\sum_{n=1}^\infty (u_n+u_{n+h})\cos(n+\tfrac{1}{2}h)x,$$

provided that $\lim_{n=\infty} u_n = 0$;

$$(b_1)\qquad \sum_1^\infty u_nU_n(x) = -\tfrac{1}{2}xu_1U_2(x) - \sum_{n=1}^\infty\left(\frac{x}{n+2}u_{n+1} + \frac{1}{2n+2}u_n\right)U_{n+2}(x),$$

provided that $\lim_{n=\infty}\frac{u_n}{2n+2}U_{n+2}(x) = 0$;

$$(c_1)\qquad \sum_1^\infty u_nX_n(x) = \frac{u_1}{3x} + \frac{2u_2}{5x}X_1(x) + \frac{1}{x}\sum_1^\infty\left(\frac{n+1}{2n+1}u_n + \frac{n+2}{2n+5}u_{n+2}\right)X_{n+1}(x),$$

* For an interesting application of these polynomials see "Integral-gleichungen, etc.," by Myller-Lebedeff; Math. Ann., vol. 64 (1907), p 390.

provided that $\lim\limits_{n=\infty} u_n X_{n+1}(x) = 0$;

$$(d_1) \qquad \sum_1^\infty u_n J_n(x) = \frac{x}{2}[u_1 J_0(x) + \tfrac{1}{2}u_2 J_1(x)] + \frac{x}{2}\sum_1^\infty \left(\frac{u_n}{n} + \frac{u_{n+2}}{n+2}\right) J_{n+1}(x),$$

provided that

$$\lim_{n=\infty} \frac{u_n}{n} J_{n+1}(x) = 0;$$

provided further that either one of the two infinite series in each equation converges.

Formulas (b_1) and (d_1) in general increase the convergence of the given series by replacing u_n by a quantity of the order of magnitude of $u_n \div n$. Formulas (a_1) and (c_1) are useful in increasing the convergence of slowly convergent alternating series. As examples we have

$$(a_2) \qquad \sum_1^\infty \frac{(-1)^{n+1}}{n} \cos nx = \frac{1}{2} + \frac{1}{2\cos\frac{1}{2}x} \sum_1^\infty \frac{(-1)^{n+1}}{n(n+1)} \cos (n + \tfrac{1}{2})x;$$

$$(b_2) \qquad \sum_1^\infty \frac{1}{n!} U_n(x) = -\frac{x}{2} U_2(x) - \sum_1^\infty \frac{2x + n + 2}{2(n+2)!} U_{n+2}(x);$$

$$(c_2) \qquad \sum_1^\infty (-1)^v X_n(x) = \frac{1}{3x} + \frac{2}{5x} X_1(x) + \frac{1}{x}\sum_1^\infty (-1)^v \frac{2n+3}{(2n+1)(2n+5)} X_{n+1}(x),$$

where $v = \frac{1}{2}n + 1$ when n is even; $v = \frac{1}{2}(n-1)$ when n is odd;

$$(d_2) \qquad \sum_1^\infty (-1)^v n J_n(x) = \frac{x}{2}[J_0(x) + J_1(x)], \ v \text{ as in } (c_2).$$

In connection with the last example it may be remarked that every series of the form

$$\sum_{n=1}^\infty n^p J_n(x),$$

where p is a positive integer, may be summed by reduction by successive steps to the series $\sum\limits_1^\infty J_n(x)$, whose sum is given in § 2 of this paper. It follows that the above series converges for all values of p and of x.

As special cases we have from (a_1) when $x = 0$ and from (c_1) when $x = 1$ respectively,

$$(a_3) \qquad \sum_{n=1}^\infty u_n = \frac{1}{2}\sum_1^h u_n + \frac{1}{2}\sum_1^\infty (u_n + u_{n+h});$$

$$(c_3) \qquad \sum_1^\infty u_n = \tfrac{1}{3}u_1 + \tfrac{2}{5}u_2 + \sum_1^\infty \left(\frac{n+1}{2n+1} u_n + \frac{n+2}{2n+5} u_{n+2}\right);$$

provided that $\lim\limits_{n=\infty} u_n = 0$, and that one of the two series in each equation converges.

Equation (a_3) is the limit of a special case under (3), namely when, for all values of n, $\psi_n = 1$, $\varphi_n = 2$, $\alpha_n = \beta_n = 1$, $r - s = h$. If with these special values we apply (3) k times in succession and if we put, symbolically,

$$\begin{aligned}(u_1 + u_2 + \cdots + u_h)(1 + u_h)^m \equiv{}& (u_1 + u_2 + \cdots + u_h) \\ &+ {}_mC_1(u_{1+h} + \cdots + u_{2h}) + {}_mC_2(u_{1+2h} + \cdots + u_{3h}) \\ &+ \cdots + {}_mC_m(u_{1+mh} + \cdots + u_{h+mh}),\end{aligned}$$

we find

$$(4)\quad \begin{aligned} s_{kh} = u_1 + u_2 + \cdots + u_{kh} ={}& \sum_{n=1}^{k} \frac{1}{2^n}(u_1 + u_2 + \cdots + u_h)(1 + u_h)^{n-1} \\ &+ \frac{1}{2^k}[s_{kh} + {}_kC_1(s_{kh} - s_h) + {}_kC_2(s_{kh} - s_{2h}) + \cdots + {}_kC_{k-1}(s_{kh} - s_{(k-1)h})]\end{aligned}$$

We shall show that the expression in the last line vanishes when $k = \infty$ provided that the series $\sum_1^\infty u_n$ converges.

The expression in question may be written in two parts,

$$\frac{s_{kh} + {}_kC_1(s_{kh} - s_h) + \cdots + {}_kC_r(s_{kh} - s_{rh})}{2^k}$$
$$+ \frac{{}_kC_{r+1}(s_{kh} - s_{(r+1)h}) + \cdots + {}_kC_{k-1}(s_{kh} - s_{(k-1)h})}{2^k}.$$

Since $\sum_1^\infty u_n$ converges, r can be taken so large that for sufficiently large values of k and for all larger values the largest of the quantities in the parentheses in the second fraction shall be less in absolute value than a pre-assigned positive quantity e. Hence the fraction itself will be less than e. Also, a positive constant M exists such that the first fraction is less in absolute value than the quantity

$$M\frac{1 + {}_kC_1 + {}_kC_2 + \cdots + {}_kC_r}{2^k}$$

and this vanishes with increasing k, no matter what the value of r.

Hence we obtain the following transformation for any *convergent* series:

$$(a_4)^* \qquad \sum_1^\infty u_n = \sum_1^\infty \frac{1}{2^n}(u_1 + u_2 + \cdots + u_h)(1 + u_h)^{n-1},$$

where h is any positive integer, and we have symbolically,

$$(1 + u_h)^m \equiv 1 + {}_mC_1u_h + {}_mC_2u_{2h} + \cdots + u_{mh}; \quad u_nu_m \equiv u_{n+m}.$$

* When $h=1$ this becomes a transformation given by L. D. Ames: "Evaluation of slowly convergent series," Annals of Mathematics, series 2, vol. 3 (1902), p. 185.

A general type of recurrent relation, of which (1) is a special case when r and s are integers, is contained in the equation,

$$(5) \qquad \varphi_n \psi_n = \sum_{i=1}^{h} \alpha_{n,i} \psi_{n+r_i},$$

h being a fixed positive integer, and $r_1, r_2, \cdots r_h$, any set of integers.

Let p be a fixed positive integer such that $p + r_i > 0$, $i = 1, 2, \cdots, h$. Also let

$$A_{n,i} = \frac{u_n}{\varphi_n} \alpha_{n,i}.$$

Then

$$\sum_{p}^{k} u_n \psi_n = \sum_{n=p}^{k} \sum_{i=1}^{h} A_{n,i} \psi_{n+r_i} = \sum_{i=1}^{h} \sum_{n=p+r_i}^{k+r_i} A_{n-r_i,i} \psi_n$$

$$= \sum_{i=1}^{h} \left[\sum_{n=p}^{k} A_{n-r_i,i} \psi_n - \sum_{n=p}^{p+r_i-1} A_{n-r_i,i} \psi_n + \sum_{n=k+1}^{k+r_i} A_{n-r_i,i} \psi_n \right],$$

where any summation whose upper limit is less than the lower is to be replaced by one in which the upper limit is at least as large as the lower according to the formula

$$\sum_{s}^{s+r-1} = \begin{cases} 0 \text{ if } r = 0, \\ -\sum_{s+r}^{s-1} \text{ if } r < 0. \end{cases}$$

With this convention the above transformation holds for negative and zero values of r_i as well as for positive values.

If now we assume that

$$\lim_{k=\infty} \sum_{i=1}^{h} \sum_{n=k+1}^{k+r_i} A_{n-r_i,i} \psi_n = 0,^*$$

we have

$$(6) \qquad \sum_{p}^{\infty} u_n \psi_n = - \sum_{i=1}^{h} \sum_{n=p}^{p+r_i-1} A_{n-r_i,i} \psi_n + \sum_{n=p}^{\infty} \left(\sum_{i=1}^{h} A_{n-r_i,i} \right) \psi_n,$$

provided that either one of the two infinite series here involved converges.

As an example we transform the series $\sum_{3}^{\infty} n^2 J_n$ by means of the recurrent relation

$$\left(\frac{2}{x}\right)^2 J_n = \frac{1}{n(n+1)} J_{n+2} + \frac{2}{n^2 - 1} J_n + \frac{1}{n(n-1)} J_{n-2}.$$

*A sufficient condition is $\lim_{n=\infty} A_{n-r_i,i} \psi_n = 0$.

Then

$$\varphi_n = \left(\frac{2}{x}\right)^2; \; h = 3; \; r_1 = 2, \; r_2 = 0, \; r_3 = -2;$$

$$\alpha_{n,1} = \frac{1}{n(n+1)}, \quad \alpha_{n,2} = \frac{2}{n^2 - 1}; \quad \alpha_{n,3} = \frac{1}{n(n-1)}.$$

Also p may be any integer greater than 2.

Equation (6) then becomes

$$\sum_3^\infty n^2 J_n = \frac{x^2}{4}\left[\tfrac{3}{2}J_1 + \tfrac{4}{3}J_2 - \tfrac{1}{2}J_3 - \tfrac{2}{3}J_4 + 4\sum_3^\infty J_n\right].$$

As a second simple example we transform the power series $\sum_h^\infty a_n x^n$ by use of the relation

$$(c_0 + c_1 x + \cdots c_{h-1}x^{h-1})x^n = c_0 x^n + c_1 x^{n+1} + \cdots + c_{h-1}x^{n+h-1}.$$

Putting

$$\varphi = c_0 + c_1 x + \cdots + c_{h-1}x^{h-1},$$

equation (6) gives

$$(6') \qquad \varphi \sum_h^\infty a_n x^n = -\sum_{i=1}^{h}\sum_{n=h}^{h+i-2} c_{i-1}a_{n-i+1}x^n + \sum_{n=h}^{\infty}\left(\sum_{i=1}^{h} c_{i-1}a_{n-i+1}\right)x^n,$$

provided that either one of the series converges. Here $c_0, c_1, \cdots, c_{h-1}$ is any arbitrary set of constants, and the general coefficient of the new series is therefore an arbitrary linear function of h successive coefficients of the given series.

If in particular we put

$$\varphi = (1 - x)^{h-1}, \quad \text{and} \quad a_n = n^p,$$

where p is a positive integer less than $h - 1$, we obtain the closed sum,

$$\sum_h^\infty n^p x^n = \frac{-1}{(1 - x)^{h-1}}\sum_{i=1}^{h}\sum_{n=h}^{h+i-2}(-1)^{i-1}_{h-1}C_{i-1}(n - i + 1)^p x^n,$$

where ${}_nC_m$ denotes the coefficient of x^m in $(1 + x)^n$.

As a check we calculate

$$4^2x^4 + 5^2x^5 + 6^2x^6 + \cdots = \frac{16x^4 - 23x^5 + 9x^6}{(1 - x)^3}, \quad |x| < 1,$$

which may be verified by clearing of fractions.

Evidently (6′) will give in closed form the sum of any convergent power series whose coefficients satisfy a known recurrent relation.

2. The ordinary functions used for developments in series satisfy,

in addition to equation (1), also a relation of the form,

$$\chi_n \psi'_n = \gamma_n \psi_{n+r} + \delta_n \psi_{n+s}, \qquad n = 1, 2, 3, \cdots, \tag{7}$$

where the quantities involved are functions of x defined for each value of n and are continuous for all the values of x in a certain interval. The prime indicates the first derivative with respect to x. We assume that χ_n does not vanish in the given interval, and that r and s are as in theorem 1.

We make use of this relation to obtain a test for the existence of a derivative of the function represented by a given series. In some examples we obtain this derivative in closed form, and an integration then gives the sum of the series.

Let the first k terms of the series be

$$S_k(x) = u_1 \psi_1 + u_2 \psi_2 + \cdots + u_k \psi_k,$$

where $u_1, u_2, \cdots u_k$ are independent of x. Then

$$S'_k(x) = \sum_1^k u_n \psi'_n = \sum_1^k \frac{u_n}{\chi_n}(\gamma_n \psi_{n+r} + \delta_n \psi_{n+s}).$$

By the same process which gave (3), this becomes

$$S'_k(x) = H(x) + G_k(x) + R_k(x),$$

where

$$H(x) = \sum_{n=1}^{r-s} \frac{\delta_n}{\chi_n} u_n \psi_{n+s},$$

$$G_k(x) = \sum_{n=1}^{k} \left(\frac{\gamma_n}{\chi_n} u_n + \frac{\delta_{n+r-s}}{\chi_{n+r-s}} u_{n+r-s}\right) \psi_{n+r}, \tag{8}$$

$$R_k(x) = - \sum_{n=k+1}^{k+r-s} \frac{\delta_n}{\chi_n} u_n \psi_{n+s}.$$

Integrating between limits c and x we have

$$S_k(x) = S_k(c) + \int_c^x H(x)dx + \int_c^x G_k(x)dx + \int_c^x R_k(x)dx.$$

From this follows at once

THEOREM 2. *Let c and x be two points of a closed interval $(c_1 c_2)$, and in this interval assume that*

(α) $\lim\limits_{k=\infty} S_k(c)$ *exists;*

(β) $G_k(x)$ *converges uniformly to a limit* $G(x)$ *when* $k = \infty$;

(γ) $\lim\limits_{k=\infty} \int_c^x R_k(x)dx = 0$, *uniformly.*

Then the series $\sum_1^\infty u_n \psi_n(x)$ *converges uniformly in* (c_1c_2) *to a function* $f(x)$ *whose derivative is*

$$f'(x) = H(x) + G(x).$$

APPLICATIONS.* We shall consider cases in which ψ_n is one of the functions X_n or J_n. Equation (7) becomes respectively

$$(x^2-1)X'_n = \frac{n(n+1)}{2n+1}(X_{n+1} - X_{n-1}); \tag{9}$$

$$2J'_n = J_{n-1} - J_{n+1}. \tag{10}$$

In the first case theorem 2 gives the following theorem on the convergence of the series $\Sigma a_n X_n$, and on the existence of a first derivative of the function represented by this series.

Let c_1 *and* c_2 *be two constants such that* $-1 < c_1 < c_2 < 1$, *and let* c *be a fixed point and* x *a variable point of the interval* (c_1c_2). *Assume that*

(α_1) *the series* $\sum_1^\infty a_n X_n$ *converges when* $x = c$;

(β_1) *the series* $\sum_1^\infty A_n X_n$ *converges uniformly in* (c_1c_2), *where*

$$A_n = \frac{n(n-1)}{2n-1}a_{n-1} - \frac{(n+1)(n+2)}{2n+3}a_{n+1};$$

(γ_1) $\lim\limits_{n=\infty} \dfrac{a_n}{\sqrt{n}} = 0$;

then the series $\sum_1^\infty a_n X_n$ *converges uniformly in* (c_1c_2) *to a function* $f(x)$ *having a continuous first derivative given by*

$$f'(x) = \frac{-2a_1}{3(x^2-1)} + \frac{1}{x^2-1}\sum_1^\infty A_n X_n.$$

For, comparing (7) and (9) we have

$$\chi_n = (x^2-1); \ \gamma_n = \frac{n(n+1)}{2n+1}; \ \delta_n = -\frac{n(n+1)}{2n+1}; \ r = 1; \ s = -1.$$

The equations (8) become

$$H(x) = \frac{-1}{x^2-1}\left(\tfrac{2}{3}a_1X_0 + \tfrac{6}{5}a_2X_1\right),$$

* Applications to trigonometric series will be found in the articles by M. Lerch and the writer already cited.

$$G_k(x) = \frac{1}{x^2-1}\sum_1^k\left(\frac{n(n+1)}{2n+1}a_n - \frac{(n+2)(n+3)}{2n+5}a_{n+2}\right)X_{n+1},$$

$$R_k(x) = \frac{1}{x^2-1}\left[\frac{(k+1)(k+2)}{2k+3}a_{k+1}X_k + \frac{(k+2)(k+3)}{2k+5}a_{k+2}X_{k+1}\right].$$

If we note that $X_0 = 1$ and merge the second term of $H(x)$ with $G_k(x)$ we may replace these two expressions respectively by

$$\bar{H}(x) = \frac{-2a_1}{3(x^2-1)},$$

$$\bar{G}_k(x) = \frac{1}{x^2-1}\sum_1^{k+1}\left(\frac{(n-1)n}{2n-1}a_{n-1} - \frac{(n+1)(n+2)}{2n+3}a_{n+1}\right)X_n.$$

By (β_1) the last sum, and hence also $G_k(x)$, approaches a limit uniformly when $k = \infty$. We next show that (γ_1) is a sufficient condition for (γ) of Theorem 2. By means of the identity

$$X_n = \frac{1}{n+1}(X'_{n+1} - xX'_n)$$

the first of the two terms forming $\int_c^x R_k(x)dx$ may be written

$$\frac{k+2}{2k+3}a_{k+1}\int_c \frac{X'_{k+1} - xX'_k}{x^2-1}dx.$$

Separating the integrand into two fractions and integrating each by parts we have, omitting the constant factor,

$$\int_c^x \frac{X'_{k+1}}{x^2-1}dx = \frac{X_{k+1}}{x^2-1}\Big]_c^x + \int_c^x \frac{2xX_{k+1}}{(x^2-1)^2}dx;$$

$$\int_c^x \frac{xX'_k}{x^2-1}dx = \frac{xX_k}{x^2-1}\Big]_c^x + \int_c^x \frac{(x^2+1)X_k}{(x^2-1)^2}dx.$$

On introducing the omitted factor, each term on the right involves either $a_{k+1}X_{k+1}$ or $a_{k+1}X_k$. Under condition (γ_1) both of these products converge uniformly to zero in (c_1c_2). For in this interval we can put X_n in the form*

$$X_n = \frac{\lambda_n + \epsilon_n}{\sqrt{n}},$$

where λ_n and ϵ_n are finite and continuous functions of x with an upper

*Brand: Les fonctions X_n de Legendre, p. 31. Bonnet: Sur le developpment des fonctions en series ordonneès suivant les fonctions X et Y.

bound independent of n. Hence a constant M exists such that

$$|\sqrt{n}X_n| < M.$$

Therefore

$$|a_n X_n| = \left|\frac{a_n}{\sqrt{n}}\sqrt{n}X_n\right| < M\left|\frac{a_n}{\sqrt{n}}\right|;$$

$$|a_{n+1}X_n| = \left|\frac{a_{n+1}}{\sqrt{n+1}}\sqrt{\frac{n+1}{n}}\sqrt{n}X_n\right| < M\sqrt{\frac{n+1}{n}}\left|\frac{a_{n+1}}{\sqrt{n+1}}\right|.$$

Hence under condition (γ_1),

$$\lim_{n=\infty} a_n X_n = 0 \quad \text{and} \quad \lim_{n=\infty} a_{n+1}X_n = 0.$$

Therefore

$$\lim_{k=\infty}\int_c^x R_k(x)dx = 0$$

uniformly in (c_1c_2), and the theorem follows.

As an example,* consider the series

$$\frac{3}{2}X_1 + \frac{7}{3\cdot 4}X_3 + \frac{11}{5\cdot 6}X_5 + \cdots$$

which converges to zero when $x = 0$. We find

$$f'(x) = \frac{1}{1-x^2} \quad \text{and} \quad f(x) = \tfrac{1}{2}\log\frac{1+x}{1-x},$$

in any interval lying between -1 and $+1$, and not reaching up to one of these points.

Passing to series of Bessel's functions, theorem 2 may be stated as follows:

Let c be a fixed point and x a variable point of any interval (c_1c_2) and assume that

(α_2) *the series* $\sum_1^\infty a_n J_n$ *converges when* $x = c$;

(β_2) *the series* $\sum_1^\infty (a_{n+2} - a_n)J_{n+1}$ *converges uniformly in* (c_1c_2);

(γ_2) $\lim_{n=\infty} a_{n+1}J_n = 0$ *uniformly in* (c_1c_2).

Then the series $\sum_1^\infty a_n J_n$ *converges uniformly in* (c_1c_2) *to a value* $f(x)$ *having a continuous first derivative given by*

$$f'(x) = \tfrac{1}{2}(a_1J_0 + a_2J_1) + \tfrac{1}{2}\sum_1^\infty (a_{n+2} - a_n)J_{n+1}.$$

*Cf. Brand, l. c., p. 99.

Here conditions (α_2), (β_2), (γ_2), are the equivalents of conditions (α), (β), (γ) of Theorem 2 respectively.

As a simple example, let $a_n = 1$, $n = 1, 2, \cdots$. Then we find that the series $\sum_1^\infty J_n(x)$ converges uniformly in any interval to a function $f(x)$ whose derivative is

$$f'(x) = \tfrac{1}{2}(J_0 + J_1).$$

Integrating from 0 to x, since the series converges to 0 when $x = 0$, and noting that

$$\int_0^x J_1 dx = 1 - J_0,$$

we have

$$f(x) = \sum_1^\infty J_n(x) = \tfrac{1}{2} - \tfrac{1}{2}J_0(x) + \tfrac{1}{2}\int_0^x J_0(x)dx.$$

Similarly,

$$\sum_1^\infty (-1)^{n+1}J_n(x) = -\tfrac{1}{2} + \tfrac{1}{2}J_0(x) + \tfrac{1}{2}\int_0^x J_0(x)dx.$$

In closing we indicate another method of utilizing a relation of the form (7), and obtain the sum of the series

$$\sum_1^\infty c^n J_n(x)$$

where c is a constant.

Let

$$S_k(x) = \sum_1^k c^n J_n(x);$$

calculating the expression in (8) we find

$$\begin{aligned} S_k'(x) &= \tfrac{1}{2}\sum_1^k (c^2 - 1)c^n J_{n+1} + \tfrac{1}{2}(cJ_0 + c^2J_1 - c^{k+1}J_k - c^{k+2}J_{k+1}) \\ &= \frac{c^2 - 1}{2c} S_k(x) + \tfrac{1}{2}(cJ_0 + J_1 - c^{k+1}J_k - c^k J_{k+1}). \end{aligned}$$

Putting $a = \dfrac{c^2 - 1}{c}$, and solving the linear differential equation of the first order for $S_k(x)$, we have, noting that $S_k(0) = 0$,

$$S_k(x) = \tfrac{1}{2}e^{ax}\int_0^x (cJ_0 + J_1)e^{-ax}dx - \tfrac{1}{2}e^{ax}\int_0^x c^k(cJ_k + J_{k+1})e^{-ax}dx.$$

If we restrict x by the inequality $|x| < h$, where h is any positive constant, the last integral approaches zero uniformly when $k = \infty$. For, noting that

$$J_k = \frac{x^k}{2^k(k+1)!}\left(1 - \frac{x^2}{2^2(k+1)} + \frac{x^4}{2^4 2!(k+1)(k+2)} - \cdots\right),$$

we can take k so large that

$$|J_k| < \frac{|x^k|}{2^k(k+1)!},$$

or so that

$$|c^k J_k| < \frac{|c^k x^k|}{2^k(k+1)!}.$$

By comparing the last fraction with the general term of the expansion of e^{cx} we see that, for all values of x, $\lim_{k=\infty} c^k J_k = 0$.

Hence the limiting form of the equation for $S_k(x)$ above is

$$\sum_1^\infty c^n J_n(x) = \tfrac{1}{2} e^{\frac{c^2-1}{c}x} \int_0^x [cJ_0(x) + J_1(x)] e^{\frac{1-c^2}{c}x} dx.$$

When $c = \pm 1$ this reduces to the preceding examples.

University of Nebraska,
September, 1910.

A THEOREM ON (m, n) CORRESPONDENCES.

BY L. I. NEIKIRK.

Emil Weyr in a paper* published in 1870 defines an n fold involution of points on a line as the points whose abscissas are roots of the equations $f(x) - \lambda\varphi(x) = 0$, where $f(x)$ and $\varphi(x)$ are rational integral functions of x, and λ is a variable parameter. He finds its double elements and other properties.

In a second paper† he introduces two involutions $f(x) - \lambda\varphi(x) = 0$, and $F(y) - \kappa\psi(y) = 0$ of degrees m and n, and terms these involutions projective when $A\lambda\kappa + B\lambda + C\kappa + D = 0$. The two involutions group the points of the two lines into sets of m and n points each, and the last equation puts these sets into a (1, 1) correspondence.

He considers the x and y as Cartesian coördinates of a point and develops the properties of the curves defined. Among other properties is the following; let the line $x = x_1$ cut the curve in the points $(x_1 : y_j)$ $j = 1, 2, \cdots n$ and the line $y = y_1$ cut the curve in the points $(x_i : y_1)$ $i = 1, 2, \cdots m$, then the curve will pass through all points of the lattice work

$$(A) \qquad x = x_i, \quad y = y_j, \quad i = 1, 2, \cdots m, \quad j = 1, 2, \cdots n.$$

The purpose of this paper is to prove the following theorem.

THEOREM. *If an (m, n) correspondence is given by a rational integral equation $f(x, y) = 0$ of degree m in x and n in y, and has a single correspondence of the type $(x_i : y_j)$, $i = 1, 2, \cdots m$, $j = 1, 2, \cdots n$, then $f(x, y) = 0$ can be reduced to the form*

$$\varphi(x) + \lambda\theta(x) = 0, \quad \psi(y) + \lambda\omega(y) = 0,$$

where φ, θ, ψ, and ω are rational integral functions, the first two of degree m and the last two of degree n, and λ is a variable parameter, and every correspondence is of the type $(x_i : y_j)$.

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*Sitzungsberichte der k. Böhmischen Gesellschaft der Wissenschaften in Prag. (1870), 14–19.
† Mathematische Annalen, III (1870), 34–44.

This theorem shows that if a rational algebraic curve $f(x, y) = 0$ passes through all points of *one such lattice work as* (A) it will pass through all points of *every such lattice work.*

Proof. Let $f(x, y) = A_0(x)y^n + A_1(x)y^{n-1} + \cdots + A_n(x)$ where the A's are rational functions of x of degree m or less.

Let

$$f(x_1, y) = a_0y^n + a_1y^{n-1} + \cdots + a_n,$$

then since

$$f(x_i, y_j) = 0, \quad i = 1, 2, \cdots m, \quad j = 1, 2, \cdots n.$$

it follows that

$$f(x_i, y) = k_i f(x_1, y),$$

and

$$A_s(x_i) = k_i a_s, \quad i = 1, 2, \cdots m, \quad s = 1, 2, \cdots n,$$

k_1 being equal to 1.

The expressions $A_s(x)/a_s$ take the m values $k_1 = 1, k_2, \cdots k_m$, when x takes the values $x_1, x_2, \cdots x_m$. The exceptional case $a_s = 0$ will be treated later.

In Lagrange's interpolation formula let

$$\varphi(x) = (x - x_1)(x - x_2)\cdots(x - x_m) \qquad [x_r \neq x_s, r \neq s],$$

$$\theta(x) = \sum_{r=1}^{r=m} \frac{k_r\varphi(x)}{(x - x_r)\varphi'(x_r)}.$$

Then $\theta(x_i) = k_i$ and the most general rational integral function of degree m which takes the m values $k_1, k_2, \cdots k_m$ when x takes the values $x_1, x_2, \cdots x_m$ is $\theta(x) + l\varphi(x)$.

Therefore

$$\frac{A_s(x)}{a_s} = \theta(x) + l_s\varphi(x), \qquad s = 1, 2, \cdots, n,$$

and

$$f(x, y) = (a_0y^n + a_1y^{n-1} + \cdots + a_n)\theta(x) + (a_0l_0y^n + a_1l_1y^{n-1} + \cdots + a_nl_n)\varphi(x)$$

$$= \psi(y)\theta(x) - \omega(y)\varphi(x)$$

If $a_s = 0$, then

$$A_s(x_i) = 0, \qquad i = 1, 2, \cdots, m,$$

so that

$$A_s(x) = l_s\varphi(x).$$

In every case therefore the correspondence is given by an equation of the form

$$\psi(y)\theta(x) - \omega(y)\varphi(x) = 0.$$

From this we get

$$\frac{\varphi(x)}{\theta(x)} = \frac{\psi(y)}{\omega(y)} = -\lambda,$$

and the theorem follows

$$\varphi(x) + \lambda\theta(x) = 0, \quad \psi(y) + \lambda\omega(y) = 0.$$

URBANA, ILLINOIS.

POINTS OF INDETERMINATE SLOPE ON THE DISCRIMINANT LOCUS OF AN ORDINARY DIFFERENTIAL EQUATION.*

By W. R. Longley.

The investigation of the solutions of an ordinary differential equation of the first order,

$$(1) \qquad f(x, y, p) = 0, \quad \left(p = \frac{dy}{dx}\right),$$

depends primarily on the theory of implicit functions. If x_0, y_0, p_0 is a set of constants satisfying equation (1); if f is an analytic function of its three arguments in the region considered; and if the first partial derivative, f_p, of f as to p does not vanish for this set of values, we may apply the fundamental existence theorem on implicit functions and obtain a solution for p of the form

$$(2) \qquad p = \varphi(x, y), \quad \text{where } p_0 = \varphi(x_0, y_0).$$

This solution is unique and φ can be expanded as a series in integral powers of $x - x_0$ and $y - y_0$. Cauchy's theorem is applicable to equation (2) and we know that through the point (x_0, y_0) in the direction determined by p_0 there passes one and only one integral curve of equation (1).

If the values x_0, y_0, p_0 satisfy not only equation (1) but also the equation

$$(3) \qquad f_p(x, y, p) = 0,$$

the fundamental theorem on implicit functions can not be applied directly, and another method is required. If p can be eliminated between equations (1) and (3) the result is the discriminant equation

$$(4) \qquad \Delta(x, y) = 0.$$

For any point (x_0, y_0) on the discriminant locus there is at least one value p_0 such that equations (1) and (3) are both satisfied. Consequently the fundamental theorem on implicit functions does not suffice for the immediate determination of all the integral curves through a point on the discriminant locus. In order to make this determination it has been necessary to place limitations on the form of the function f. Supposing f to be a rational integral function of y and p, Hamburger† has based his

* Read before the American Mathematical Society, December 28, 1910.

† Crelle's Journal, vol. 112 (1893), pp. 205–246. See also Schlesinger, Differentialgleichungen, Sammlung Schubert, Zweite Auflage, pp. 238–281, and Horn, Differentialgleichungen, S.S., pp. 348–356.

investigations on the known theory of algebraic functions of two variables, and, consequently, has excluded from consideration any point (x_0, y_0) for which equation (1) is satisfied identically in p. The principal object of this paper is an investigation of the integral curves through such a singular point. The first part is devoted to a consideration of equations of the second degree in p. In the latter part an extension is made under certain conditions to equations which are of higher degree, or even transcendental, in p.

It is convenient to recall here, for reference later, a result due to Briot and Bouquet.* They have studied an equation of the form $xp = \varphi(x, y)$, where φ can be expanded as a power series in x and y, and $\varphi(0, 0) = 0$. The term of the first power in y being of prime importance, the equation is written

$$xp - ay = a_{10}x + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots. \tag{5}$$

Assuming a solution of the form

$$y = b_1x + b_2x^2 + \cdots, \tag{6}$$

the following results are obtained.

i. If a is not a positive integer, the coefficients b_i are uniquely determined, and there is one and only one solution of the type (6), that is representable as a series in integral powers of x.

ii. If $a = 1$, $a_{10} \neq 0$, there is no solution of the type (6).

iii. If $a = 1$, $a_{10} = 0$, the coefficient b_1 is arbitrary, while the remaining coefficients are uniquely determined in terms of the first one. A singly infinite number of integral curves with arbitrary slope pass through the origin.

iv. If $a = q$, where q is a positive integer greater than unity, it follows directly by substituting the value of y from equation (6) in equation (5) that the coefficients $b_1, \cdots b_{q-1}$ are uniquely determined. The equation for the determination of b_q either leads to a contradiction, that is, no solution of the type (6) exists, or is satisfied for an arbitrary value of b_q. In the latter case the remaining coefficients are uniquely determined in terms of b_q. A singly infinite number of integral curves all having the same slope pass through the origin.

I.

Consider the equation

$$f(x, y, p) \equiv A(x, y)p^2 + 2B(x, y)p + C(x, y) = 0. \tag{7}$$

In the region considered the coefficients A, B, and C are supposed to be

* Journal de l'école polytechnique, vol. 21 (1856), p. 161.

analytic functions of x and y, and equation (7) is supposed to be irreducible in x, y, and p. The discriminant equation is

$$\Delta(x, y) \equiv B^2 - AC = 0. \tag{8}$$

Let $y = \eta(x)$ be a solution of equation (8), where η is an analytic function of x in a certain neighborhood of the value $x = c$. We suppose that $A[x, \eta(x)] \not\equiv 0$, and, in particular, $A[c, \eta(c)] \neq 0$.*

From equation (7)

$$p = \frac{-B + \sqrt{\Delta}}{A}. \tag{9}$$

For any point not on the discriminant locus there are two distinct values of p. For any point on the curve $y = \eta(x)$ equation (7) has a double root

$$p = \zeta(x) = -\frac{B[x, \eta(x)]}{A[x, \eta(x)]}. \tag{10}$$

Now A, B, and C may be expanded as power series in $y - \eta$:

$$A(x, y) = A(x, \eta) + a_1(y - \eta) + \cdots,$$
$$B(x, y) = B(x, \eta) + b_1(y - \eta) + \cdots,$$
$$C(x, y) = C(x, \eta) + c_1(y - \eta) + \cdots,$$

where the coefficients are analytic functions of x. Since $\Delta[x, \eta(x)] \equiv 0$, the expansion for Δ has the form

$$\Delta(x, y) = (y - \eta)^a[d_0 + d_1(y - \eta) + \cdots],$$

where $d_0(x) \not\equiv 0$, and we suppose, in particular, $d_0(c) \neq 0$.†
Substituting these expansions in equation (9) gives

$$p = \frac{-B(x, \eta) - b_1(y - \eta) - \cdots + (y - \eta)^{a/2}[d_0 + d_1(y - \eta) + \cdots]^{\frac{1}{2}}}{A(x, \eta)\left[1 + \dfrac{a_1}{A(x, \eta)}(y - \eta) + \cdots\right]}.$$

Since $A \neq 0$, $d_0 \neq 0$, the second member of this equation may be expanded in powers of $(y - \eta)^{\frac{1}{2}}$, giving

$$p - \zeta(x) = g_0(y - \eta)^{\frac{k}{2}} + g_1(y - \eta)^{\frac{k+1}{2}} + \cdots, \tag{11}$$

* If A vanishes for $x = c$, $y = \eta(c)$, while C does not vanish at this point, the problem is treated by considering the reciprocal of p.

† For a study of the case $d_0(c) = 0$ see a paper by Petrovitch, Mathematische Annalen, vol. 50 (1898), p. 103.

where k is a positive integer and the coefficients g_i are analytic functions of x in the neighborhood of $x = c$. There are now two cases to be considered.

Case I.—If $d\eta/dx \equiv \zeta$, then $y = \eta(x)$ is a solution of the given differential equation. In this case equation (11) may be written

$$\frac{d(y-\eta)}{dx} = g_0(y-\eta)^{\frac{k}{2}} + g_1(y-\eta)^{\frac{k+1}{2}} + \cdots.$$

By making the substitution $y - \eta = u^2$, this equation becomes

$$2u\frac{du}{dx} = g_0u^k + g_1u^{k+1} + \cdots. \tag{12}$$

The factor u may be canceled since $u = 0$ corresponds to the solution $y = \eta(x)$ already known. Cauchy's existence theorem is applicable to the remaining equation and asserts that there exists a unique solution for u vanishing with $x = c$.

If $k = 1$ this solution has the form

$$u = u_1(x-c) + u_2(x-c)^2 + \cdots,$$

where the coefficients u_i are not all zero. In this case we see that through the point $P[c, \eta(c)]$ besides the integral curve $S\colon y = \eta$, there passes one other integral curve

$$y = \eta(x) + u_1^2(x-c)^2 + \cdots.$$

This integral curve is tangent to S at the point P. Since P is any point on the curve S for which the hypotheses enumerated above are satisfied, it follows that S is an envelope of integral curves, that is, a *singular* solution.

If $k > 1$ the only solution of equation (12) is $u = 0$. In this case S is the only integral curve passing through P, and $y = \eta(x)$ is merely a *particular* solution.

Case II.—If $\frac{d\eta}{dx} \mathrel{\not\equiv} \zeta$ we suppose, in particular, that

$$\zeta - \frac{d\eta}{dx} = \gamma \neq 0, \text{ for } x = c.$$

Equation (11) may be written

$$\frac{d(y-\eta)}{dx} = \zeta - \frac{d\eta}{dx} + g_0(y-\eta)^{\frac{k}{2}} + \cdots.$$

By making the substitution $y - \eta = u^2$ this equation becomes

$$2u\frac{du}{dx} = \zeta - \frac{d\eta}{dx} + g_0u^k + \cdots,$$

whence

$$\frac{dx}{du} = \frac{2u}{\zeta - \dfrac{d\eta}{dx} + g_0 u^k + \cdots}.$$

Since the denominator does not vanish for $x = c$, $u = 0$, the second member may be expanded in powers of $x - c$ and u, containing u as a factor:

$$\frac{dx}{du} = \frac{2}{\gamma} u + uP, \tag{13}$$

where P is a power series vanishing with $x = c$, $u = 0$. Equation (13) admits a unique solution for $x - c$ as a power series in u, and this series contains u^2 as a factor:

$$x - c = \frac{1}{\gamma} u^2 + \cdots.$$

Reverting this series we get

$$u = \sqrt{y - \eta} = \sqrt{\gamma}(x - c)^{\frac{1}{2}} + \cdots,$$

and, squaring, the solution of the original differential equation is found in the form

$$y - \eta = \gamma(x - c) + \gamma_1(x - c)^{\frac{3}{2}} + \cdots.$$

From this it appears that through the point $P[c, \eta(c)]$ there passes one integral curve of equation (7), and this curve has a cusp at P. Since $\gamma \neq 0$ the cuspidal tangent does not coincide with the tangent to the curve $y = \eta(x)$. In this case the branch $y = \eta$ of the discriminant curve is a locus of cusps on integral curves.

The conclusions reached so far may be summarized as follows: If Γ: $y = \eta(x)$ is an analytic branch of the discriminant locus three cases may occur. (1) $y = \eta(x)$ may be a particular solution. Through a general point on Γ there passes *no* other integral curve. (2) $y = \eta(x)$ may be a singular solution. Through a general point P on Γ there passes *one* other integral curve and it is tangent to Γ at P. (3) $y = \eta(x)$ may be a cusp locus. Through a general point P on Γ there passes *one* integral curve and it has a cusp at P. In general the cuspidal tangent does not coincide with the tangent to Γ at P.

The preceding results are due to Hamburger. Certain points have been excluded by the hypotheses made during the argument, in particular, those points for which A, B, and C all vanish. Suppose now that for a certain point, which will be taken for the origin, A, B, and C vanish. Let α be the degree of the terms of lowest degree in A, B, or C. Then

$$A(x, y) = a_{\alpha,0}x^\alpha + a_{\alpha-1,1}x^{\alpha-1}y + \cdots + a_{0,\alpha}y^\alpha + a_{\alpha+1,0}x^{\alpha+1} + \cdots,$$

$$B(x, y) = b_{\alpha,0}x^\alpha + \cdots, \quad C(x, y) = c_{\alpha,0}x^\alpha + \cdots.$$

By making the substitution

$$y = xv, \quad p = xv' + v, \quad \left(v' = \frac{dv}{dx}\right) \tag{14}$$

in equation (7), and canceling the factor x^α, we get

$$A_1(xv' + v)^2 + 2B_1(xv' + v) + C_1 = 0, \tag{15}$$

where

$$A_1(x, v) = a_{\alpha,0} + a_{\alpha-1,0}v + \cdots + a_{0,\alpha}v^\alpha + a_{\alpha+1,0}x + \cdots,$$

$$B_1(x, v) = b_{\alpha,0} + \cdots, \quad C_1(x, v) = c_{\alpha,0} + \cdots,$$

are power series in x and v which converge for all values of v, if x is small enough.

A first necessary condition for an analytic integral curve through the origin is that for $x = 0$, v satisfies the equation

$$A_1(0, v)v^2 + 2B_1(0, v)v + C_1(0, v) = 0. \tag{16}$$

This is an algebraic equation of degree $\alpha + 2$ if $a_{0,\alpha} \neq 0$. The initial value of v is the slope of the integral curve at the origin. Hence there are in general $\alpha + 2$ critical slopes which the integral curve may conceivably have. From equation (15) we get

$$xv' = \frac{-A_1v - B_1 + \sqrt{\Delta_1}}{A_1}, \tag{17}$$

where $\Delta_1 = B_1^2 - A_1C_1$. The following cases are presented.

Case I.—A critical slope v_1 not tangent to a branch of the discriminant locus. In this case

$$B_1^2(0, v_1) - A_1(0, v_1)\, C_1(0, v_1) \neq 0,$$

and, if $A_1(0, v_1) \neq 0$, the second member of equation (17) may be expanded in powers of x and $v - v_1$, and the equation is of the form (5) investigated by Briot and Bouquet. There may be no analytic solution or there may be one of the form

$$v - v_1 = b_1x + b_2x^2 + \cdots,$$

where the coefficients are either uniquely determined or contain an arbitrary constant.

The solution of equation (7) becomes

$$y = v_1x + b_1x^2 + \cdots.$$

Hence through the singular point in the direction determined by v_1 there

may be either (1) no analytic integral curve, or (2) one, or (3) a singly infinite number.

Case II.—A critical slope v_1 tangent to an analytic branch of the discriminant curve at the origin. The equation of the branch of the discriminant curve may be written in the form $y = \eta(x) = x\psi$, where ψ is a power series in x. From equation (10)

$$p = \zeta(x) = -\frac{B_1(x, \psi)}{A_1(x, \psi)}.$$

For a singular or particular solution

(18) $$x\psi'(x) + \psi(x) \equiv \zeta(x).$$

For a cusp locus

(19) $$x\psi'(x) + \psi(x) \not\equiv \zeta(x),$$

but we might have

(20) $$\psi(0) = \zeta(0).$$

In this case Δ_1 vanishes for $x = 0$, $v = v_1$ and, if $A_1(0, v_1) \neq 0$, the second member of equation (17) can be expanded in powers of $(v - \psi)^{\frac{1}{2}}$ with coefficients which are analytic functions of x, and the result can be written

$$x\frac{d(v-\psi)}{dx} = -\psi - x\psi' + \zeta + g_0(v-\psi)^{\frac{k}{2}} + g_1(v-\psi)^{\frac{k+1}{2}} + \cdots.$$

By putting $v - \psi = u^2$ this equation becomes

(21) $$2xu\frac{du}{dx} = -\psi - x\psi' + \zeta + g_0u^k + g_1u^{k+1} + \cdots.$$

(*a*) If $y = x\psi(x)$ is a solution of equation (7) then the relation (18) shows that a factor u may be canceled from equation (21), leaving the equation of Briot and Bouquet. There may be no analytic solution or there may be one of the form

$$u = b_1x + b_2x^2 + \cdots,$$

where the coefficients are either uniquely determined or contain an arbitrary constant. The corresponding solution of equation (7) is

$$y = x\psi(x) + b_1^2x^3 + \cdots.$$

Hence if $y = x\psi(x)$ is a singular or particular solution of equation (7) there may be either (1) no other integral curve tangent to it at the singular point, or (2) one, or (3) a singly infinite number. (*b*) If $y = x\psi(x)$ is a cusp locus then relation (19) shows that a factor u can not be canceled from equation (21). In general the second member does not vanish for $x = 0$, $u = 0$, and there is no analytic solution. However, if the relation (20) holds, analytic solutions may exist. The equation is similar to equation (5), and may have

one or an infinite number of solutions. Hence if $y = x\psi(x)$ is a cusp locus, there is, in general, no integral curve tangent to it at the singular point; but there may be one (see example IV), or even an infinite number (see example V).

Case III.—A critical slope tangent to a non-analytic branch of the discriminant locus. This case is not treated in the present paper.

Case IV.—It may happen that equation (16) is satisfied identically in v, and hence the initial slope is arbitrary. Then a factor x may be canceled from both sides of equation (17) and Cauchy's theorem is applicable to the resulting equation for all initial values of v except a certain finite number for which $A_1 = 0$ or $\Delta_1 = 0$. Hence there are an infinite number of integral curves passing through the singular point, the slope being arbitrary, except for a finite number of directions which require special investigation. See examples I*b* and IV.

Example I.

$$(22)\qquad \begin{aligned} f(x, y, p) \equiv x^3(1+x)^2p^2 + 2xy(1+x)[2y - x(3+4x)]p \\ + y^2[x(3+4x)^2 - 4y(2+3x)] = 0, \end{aligned}$$

$$\Delta(x, y) = 4x^2(1+x)^2y^3(y - x - x^2) = 0.$$

There are two analytic branches:

$$y = \eta(x) = 0, \text{ a particular solution,}$$

$$y = \eta(x) = x + x^2, \text{ a singular solution.}$$

(*a*) Equation (22) has a singular point at the origin, which is a point of intersection of the two analytic branches of the discriminant curve. For this example equation (17) becomes

$$(23)\qquad xv' = \frac{v(2+3x) - 2v^2 + 2\sqrt{v^3[v - (1+x)]}}{1+x}.$$

There are two initial values of v, each falling under case II, namely, 0 and 1, and for each of these values the quantity under the radical vanishes.

Taking first the initial value $v = 0$, expanding the second member of equation (23) in powers of $v^{\frac{1}{2}}$ and setting $v = u^2$, we get (corresponding to equation (21))

$$2xu\frac{du}{dx} = 2u^2 + x(1+x)^{-1}u^2 + u^3P,$$

where P denotes a power series in x and u. Canceling the factor u which corresponds to the solution $y = 0$ already known, this equation becomes

$$x\frac{du}{dx} - u = \tfrac{1}{2}x(1+x)^{-1}u + u^2P. \tag{24}$$

Equation (24) is of the form (5) and admits a solution

$$u = b_1x + b_2x^2 + \cdots,$$

where b_1 is arbitrary. The corresponding solution of equation (7) is

$$y = b_1^2x^3 + \cdots.$$

Hence an infinite number of integral curves pass through the singular point, all tangent to the particular solution $y = 0$.

Taking next the initial value $v = 1$, expanding the second member of equation (23) in powers of $(v - \psi)^{\frac{1}{2}}$ $[\psi = 1 + x]$, and setting $v - \psi = u^2$, we get

$$2xu\frac{du}{dx} = 2u + uQ,$$

where Q denotes a power series in x and u, vanishing with x and u. Canceling the factor u which corresponds to the solution $y = x + x^2$ already known, it is seen that the second member does not vanish with x and u. Hence there exists no other analytic solution, that is, through this point there passes no integral curve tangent to the singular solution.

(b) Equation (22) has a singular point at $(-1, 0)$. Translating the origin to this point the equation becomes

$$\begin{aligned} f(x, y, p) \equiv x^2(x-1)^3p^2 &+ 2xy(x-1)[2y - (x-1)(4x-1)]p \\ &+ y^2[(x-1)(4x-1)^2 - 4y(3x-1)] = 0. \end{aligned} \tag{25}$$

The discriminant equation becomes

$$\Delta(x, y) = 4x^2(x-1)^2y^3[y + x - x^2] = 0.$$

There are two analytic branches passing through the origin:

(a) $\quad y = x\psi(x) = 0$, a particular solution,

(b) $\quad y = x\psi(x) = x(-1 + x)$, a singular solution.

For this example equation (17) becomes

$$xv' = \frac{3x(x-1)v - 2xv^2 + 2xv\sqrt{v^2 - v(x-1)}}{(x-1)^2}. \tag{26}$$

The initial value of v is arbitrary. When $x = 0$ the denominator does not vanish for any value of v, but the quantity under the radical vanishes for $v = 0$, $v = -1$. Canceling the factor x, the second member of equation (26) may be expanded in powers of x and $v - v_1$ where v_1 is arbitrary

except $v_1 \neq 0$, $v_1 \neq -1$. Cauchy's theorem is applicable to this equation and there exists a unique solution

$$v - v_1 = b_1x + b_2x^2 + \cdots.$$

The corresponding solution of equation (25) is

$$y = v_1x + b_1x^2 + \cdots.$$

Hence in every direction not tangent to a branch of the discriminant curve there passes *one* integral curve.

To examine the initial value $v = 0$ we expand the second member of equation (26) in powers of x and $v^{\frac{1}{2}}$. After making the substitution $v = u^2$ a factor u, corresponding to the solution $y = 0$ already known, may be canceled, and the resulting equation is

$$\frac{du}{dx} = uP,$$

where P denotes a power series in x and u. By Cauchy's theorem this equation admits a unique solution which is evidently $u = 0$. Hence $y = 0$ is the only integral curve through this point with slope equal to zero.

To examine the initial value $v = -1$ we expand the second member of equation (26) in powers of $(v - \psi)^{\frac{1}{2}}$, $[\psi = -1 + x]$, with coefficients which are analytic functions of x. After making the substitution $v - \psi = u^2$, a factor u, corresponding to the solution $y = -x + x^2$ already known, may be canceled, and the resulting equation is

$$\frac{du}{dx} = \frac{1}{\sqrt{x-1}} + uP.$$

By Cauchy's theorem this equation admits a unique solution

$$u = b_1x + b_2x^2 + \cdots.$$

The corresponding solution of equation (25) is

$$y = -x + x^2 + b_1^2x^3 + \cdots.$$

Hence there is *one* other integral curve tangent to the singular solution at this point.

The results concerning equation (22) may now be summarized as follows.* Through any point not on the discriminant locus there are *two* integral curves with distinct tangents. Through any point on the

* It is understood that the direction of the Y-axis is tacitly excluded. Also the branches $x=0$, $x=-1$ of the discriminant locus are excluded. They can not be represented by equations of the form $y=\eta(x)$.

branch $y = 0$ of the discriminant locus, except the singular points $O(0, 0)$ and $P(-1, 0)$, there passes only *one* integral curve, namely $y = 0$. Through any point on the branch $y = x + x^2$ of the discriminant locus there are two integral curves having the same direction. Through the singular point O there is one integral curve (the singular solution) with slope equal to unity, and an infinite number with slope equal to zero. Through the singular point P, in addition to the singular integral, there is *one* integral curve in every direction. The general solution of equation (22) is

$$y = c^2x^3\frac{1 + x}{2cx - 1}.$$

Example II.

$$f(x, y, p) = Ap^2 + 2Bp + C = 0, \tag{27}$$

where

$$A = (x + y)^3 + (x - y)(2x + y)^2,$$

$$B = (x + y)^3 - (x - y)(2x + y)(x + 2y),$$

$$C = (x + y)^3 + (x - y)(x + 2y)^2.$$

$$\Delta(x, y) = 9(y + x)^5(y - x).$$

Of the two analytic branches of the discriminant locus, $y = -x$ is a particular solution and $y = x$ is a cusp locus. The origin is the only singular point. At any other point on the cusp locus the cuspidal tangent is perpendicular to the line $y = x$. For this example equation (17) becomes

$$xv' = \frac{1 + v}{5 + 3v}\{4 - 3(1 + v)^2 + 3\sqrt{(1 + v)^3(v - 1)}\}. \tag{28}$$

There are three initial values of v, namely,

$$v_1 = \tfrac{1}{3}\sqrt{-3}, \quad v_2 = -\tfrac{1}{3}\sqrt{-3}, \quad v_3 = -1.$$

Expanding the second member in powers of $v - v_1$, equation (28) becomes

$$x\frac{d(v - v_1)}{dx} = g_0(v - v_1) + (v - v_1)^2P,$$

where

$$g_0 = \frac{(\sqrt{-3} - 9)(\sqrt{3} + \sqrt{-1})}{2\sqrt{3}(5 + \sqrt{-3})}.$$

This is the equation of Briot and Bouquet admitting a unique solution which is seen to be $v - v_1 = 0$. The corresponding solution of equation (27) is $y = \frac{1}{3}\sqrt{-3}x$. Expanding in powers of $v - v_2$ it is shown in a similar manner that the only solution is $v - v_2 = 0$, and the corresponding solution

of equation (27) is $y = -\frac{1}{3}\sqrt{-3}\,x$. These two integrals can be combined into a single one and written in the form $3y^2 + x^2 = 0$.

For the initial value $v_3 = -1$ the second member must be expanded in powers of $(v+1)^{\frac{1}{2}}$, and equation (28) becomes

$$x\frac{d(v+1)}{dx} = 2(v+1) + (v+1)^{\frac{3}{2}}P, \tag{29}$$

where P denotes a power series in $(v+1)^{\frac{1}{2}}$. By setting $v + 1 = u^2$ equation (29) takes the form

$$xu\frac{du}{dx} = u^2 + \tfrac{1}{2}u^3P.$$

Canceling the factor u there is left the equation of Briot and Bouquet admitting a solution of the form

$$u = b_1x + b_2x^2 + \cdots,$$

where b_1 is arbitrary. The corresponding solution of equation (27) is

$$y = -x + b_1^2x^3 + \cdots.$$

Hence in addition to the curve $3y^2 + x^2 = 0$ which has a conjugate point at the origin, there are an infinite number of integral curves passing through this singular point each one of which is perpendicular to the cusp locus. There is no integral curve having a cusp at the origin.

The general integral of equation (27) is

$$(x+y)(x+y-c)^2 + (x-y)^3 = 0.$$

III.

Consider now an equation of more general form

$$f(x, y, p) = 0. \tag{30}$$

To take up only the simplest case suppose that f can be expanded as a power series in $(x-a)$, $(y-b)$, and $(p-c)$, and that the following conditions hold.

$$f(a, b, c) = 0,\quad f_p(a, b, c) = 0,\quad f_y(a, b, c) \neq 0,\quad f_{pp}(a, b, c) \neq 0. \tag{31}$$

With this hypothesis the fundamental theorem on implicit functions may be applied to solve the equations $f = 0$, $f_p = 0$ for

$$y = \eta(x), \quad \text{where } b = \eta(a), \tag{32}$$

$$p = \zeta(x), \quad \text{where } c = \zeta(a). \tag{33}$$

Equation (32) is the discriminant locus and equation (33) gives the value of p (near $p = c$) at any point on this locus. For every value of x within a certain neighborhood of $x = a$ the following conditions are satisfied,

$$f[x, \eta(x), \zeta(x)] = 0, \quad f_p[x, \eta(x), \zeta(x)] = 0,$$

$$f_y[x, \eta(x), \zeta(x)] \neq 0, \quad f_{pp}[x, \eta(x), \zeta(x)] \neq 0.$$

Hence equation (30) may be written in the form

$$f = 0 = f_{10}(y - \eta) + f_{20}(y - \eta)^2 + f_{11}(y - \eta)(p - \zeta) + f_{02}(p - \zeta)^2 + \cdots,$$

where the coefficients are analytic functions of x. Since f_{10} and f_{02} do not vanish in the region considered this equation may be solved for $p - \zeta$ as a series in $(y - \eta)^{\frac{1}{2}}$:

$$p - \zeta = g_0(y - \eta)^{\frac{1}{2}} + g_1(y - \eta) + \cdots.$$

This equation is of the form (11) where $k = 1$. Therefore, if

$$\frac{d\eta}{dx} \equiv \zeta, \quad y = \eta(x)$$

is a singular solution; and, if

$$\frac{d\eta}{dx} \not\equiv \zeta, \quad y = \eta(x)$$

is a cusp locus.

Suppose now that for $x = 0$, $y = 0$ equation (30) is satisfied identically in p. Then it may be written in the form

$$f(x, y, p) = A(x, y) + B(x, y)p + \cdots = 0,$$

where, as before,

$$A(x, y) = a_{\alpha, 0}x^{\alpha} + \cdots + a_{0, \alpha}y^{\alpha} + a_{\alpha+1, 0}x^{\alpha+1} + \cdots, \quad B(x, y) = b_{\alpha, 0}x^{\alpha} + \cdots.$$

Making the substitution (14) and canceling a factor x^{α}, gives

(34) $$A_1(x, v) + B_1(x, v)(xv' + v) + \cdots = 0.$$

The initial values of v are determined by the condition

(35) $$A_1(0, v) + B_1(0, v)v + \cdots = 0.$$

Suppose $v = c$ satisfies condition (35). Then equation (34) may be written in the form

(36) $$A_2(x, v - c) + B_2(x, v - c)(xv' + v - c) + \cdots = 0,$$

where A_2, B_2, $\cdots$ are power series in x and $v - c$, and A_2 contains no constant term. If the initial slope $v = c$ is not tangent to a branch of the discriminant locus, that is, if B_2 does not vanish for $x = 0$, $v = c$, the fundamental theorem on implicit functions is applicable to equation (36) which may be solved for the quantity $xv' + v - c$ as a power series in x and $v - c$. Hence we are led to the equation of Briot and Bouquet with its known results. If the initial slope $v = c$ is tangent to a branch of the discriminant locus, the fundamental theorem on implicit functions gives no information concerning the solution. In special cases it is possible to solve equation (36) for $xv' + v - c$ as a series involving fractional powers, and the results may be investigated by the methods used in the preceding examples.

Example III.

$$f(x, y, p) = x\sqrt{1 - p^2} - y \operatorname{arc} \cos p = 0. \tag{37}$$

$$\Delta(x, y) = \frac{y}{x} - \cos\frac{\sqrt{x^2 - y^2}}{y} = 0.$$

The discriminant locus consists of an infinite number of straight lines passing through the origin, namely, of all the tangent lines which can be drawn from the origin to the curve $y = \sin x$. Any one of the branches is given by

$$y = x \cos m,$$

where m is a constant satisfying the equation

$$m - \tan m = 0. \tag{38}$$

The origin is a singular point to be investigated later. If m is any value, except 0, which satisfies equation (38), then conditions (31) are satisfied for

$$a \neq 0,\ b = a \cos m,\ c = \cos m.$$

Also, corresponding to equations (32) and (33),

$$y = \eta(x) = x \cos m, \quad p = \zeta(x) = \cos m.$$

Hence $y = x \cos m$ is a singular solution.

To investigate the singular point at the origin we have, corresponding to equation (35),

$$\sqrt{1 - p^2} - v \operatorname{arc} \cos p = 0. \tag{39}$$

This determines an infinite number of initial values v_1, namely, $v_1 = \cos m$, where m satisfies equation (38). The value $m = 0$ ($v_1 = 1$) requires special investigation. For any other initial value equation (39) may be solved for p as a series in $(v - v_1)^{\frac{1}{2}}$. The result is

$$x\frac{d(v-v_1)}{dx} = \sqrt{-2m\sin m}(v-v_1)^{\frac{1}{2}} + \cdots.$$

By putting $v - v_1 = u^2$ it is apparent that this equation admits no analytic solution (except $v - v_1 = 0$).

To investigate the initial value $v_1 = 1$ we expand arc cos p in powers of $(1 - p)^{\frac{1}{2}}$.

$$\text{arc}\cos p = \sqrt{2}\sqrt{1-p}R,$$

where

$$R = 1 + \tfrac{1}{12}(1-p) + \cdots$$

is a power series in $1 - p$. Hence the factor $\sqrt{1-p}$ may* be canceled from equation (39), and the remaining equation can be solved for $p - 1$ as a series in $v - 1$. We have to consider then

$$(40) \qquad x\frac{d(v-1)}{dx} - 2(v-1) = (v-1)^2P,$$

where P is a series in $v - 1$. This is the equation of Briot and Bouquet admitting a solution of the form

$$v - 1 = b_2x^2 + b_3x^3 + \cdots,$$

where b_2 is arbitrary. The corresponding solution of equation (37) is

$$y = x + b_2x^3 + b_3x^4 + \cdots.$$

Hence an infinite number of integral curves pass through the origin tangent to the branch $y = x$ of the discriminant locus. Through any other point on this line it is evident, by applying Cauchy's theorem to equation (40), that there is no other integral curve. There is no integral curve through the origin tangent to any other branch of the discriminant locus.

The general solution of equation (37) is

$$y = c\sin\frac{x}{c}.$$

Every curve of this family passes through the origin with slope equal to unity. If the branches of the discriminant locus are ordered according to increasing values of m, then, as x increases, every curve of the general integral touches each branch of the envelope in order.

Example IV.

$$x^4p^2 - 4xy^2p + 4y^3 = 0.$$

* Corresponding to this factor there is the solution $y = x$ of equation (37). This equation has the property of a reducible algebraic equation. It is apparent by inspection that $y = x + constant$ is an integral.

Example V.

$$18(x - y)(xp - y)^2 + (x + y)^5(p + 1)^2 = 0.$$

Example VI.

$$4(x - y)[(4x - y)p + x - 4y]^2 + (x + y)^7(p + 1)^2 = 0.$$

Example VII.

$$x\sqrt{1 - p^2} + y \text{ arc } \sin p = 0.$$

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BOUNDARY PROBLEMS AND GREEN'S FUNCTIONS FOR LINEAR DIFFERENTIAL AND DIFFERENCE EQUATIONS.

By Maxime Bôcher.

In Part I of the present paper I have established some simple results of a general character concerning linear boundary problems in one dimension. These results were established by Mason* for the equation of the second order by a method which admits of easy generalization. The proof here given of the main result (Theorem 3) is, however, of an even more elementary character than that given by Mason.

Part II is devoted to Green's Functions in one dimension. This conception was introduced by Burkhardt† for the special differential equation $y'' = 0$ (Laplace's Equation in one dimension), and extended by the present writer to the general linear differential equation of the nth order.‡ No proofs were given in this paper since it was thought, apparently erroneously, that the methods used would be sufficiently obvious. Subsequent writers, notably Westfall,§ established some of these results together with some similar ones for certain other boundary conditions, but, owing to their failure to appreciate the statement clearly made in my note just referred to that the Green's function will in general satisfy *different* boundary conditions when regarded as a function of one of its arguments from those which it satisfies when regarded as a function of the other, they failed to consider the general case. This general case, which includes as a special case the boundary conditions of my note, was first explicitly considered, though without proofs, by Birkhoff.‖ Still more recently an exhaustive treatment of the subject for systems of linear differential equations of the first order has been given by Bounitzky,** who also applies his results at length to the equation of the nth order. In this paper the earlier literature is fully cited with the exception of Birkhoff's paper, which really contains the whole theory in brief so far as the single equation of the nth order goes. In Part II of the present paper I publish for the first time, along with a small

* Trans. Amer. Math. Soc., vol. 7 (1906), p. 339.

† Bull. de la Soc. Math. de France, vol. 22 (1894), p. 71.

‡ Bull. Amer. Math. Soc., second series, vol. 7 (1901), p. 297.

§ Dissertation, Göttingen (1905).

‖ Trans. Amer. Math. Soc., vol. 9 (1908), p. 377.

** Liouville's Journal, 6th series, vol. 5 (1909), p. 65.

amount of additional matter, the method of treatment I had in mind ten years ago, modified only to the slight extent necessary to make it apply to the general linear boundary conditions.

Finally in Part III it is shown briefly how the results of Parts I and II can be carried over to the case of linear *difference* equations. The details are, in the main, fairly obvious, and are therefore to a considerable extent suppressed.

I. Boundary Problems for Ordinary Linear Differential Equations.

1. Let us consider the differential equation

$$\frac{d^n u}{dx^n} + p_1 \frac{d^{n-1}u}{dx^{n-1}} + \cdots + p_n u = p, \tag{1}$$

where the p's are continuous but not necessarily real functions of the real variable x when $a \leqq x \leqq b$, together with certain boundary conditions which we will indicate as follows:

If φ is a function of x which at the points a and b has derivatives of the first $n - 1$ orders, we write

$$\begin{aligned} A_i(\varphi) &\equiv \alpha_i \varphi(a) + \alpha_i' \varphi'(a) + \cdots + \alpha_i^{[n-1]} \varphi^{[n-1]}(a), \\ B_i(\varphi) &\equiv \beta_i \varphi(b) + \beta_i' \varphi'(b) + \cdots + \beta_i^{[n-1]} \varphi^{[n-1]}(b), \end{aligned} \qquad (i = 1, 2, \cdots n) \tag{2}$$

where the α's and β's are given constants. If now we let

$$W_i(\varphi) \equiv A_i(\varphi) + B_i(\varphi), \tag{3}$$

we may write our boundary conditions as follows, the γ's being given constants:

$$W_i(u) = \gamma_i \qquad (i = 1, 2, \cdots n). \tag{4}$$

The problem of determining all solutions of (1) which satisfy conditions (4) we speak of for brevity as the problem (1), (4); and this problem we call *homogeneous* only when $p \equiv 0$ and all the constants γ_i vanish. Two cases of some importance may be designated as *semi-homogeneous* problems, namely that in which $p \equiv 0$ but at least one γ_i is not zero, and also that in which all the γ_i's vanish but p is not identically zero.

The homogeneous equation

$$\frac{d^n u}{dx^n} + p_1 \frac{d^{n-1}u}{dx^{n-1}} + \cdots + p_n u = 0 \tag{5}$$

is called the *reduced equation* when considered in connection with the non-homogeneous equation (1), and the homogeneous boundary conditions

$$W_i(u) = 0 \qquad (i = 1, 2, \cdots n) \tag{6}$$

we will call the *reduced boundary conditions* when we consider them in connection with (4). The homogeneous system (5), (6) we may similarly speak of as the *reduced system.*

DEFINITION. *The homogeneous system* (5), (6) *is said to be incompatible if* (5) *has no solution except zero which satisfies* (6). *It is said to have k-fold compatibility if* (5) *has k and only k linearly independent solutions which satisfy* (6).

If (5), (6) has k-fold compatibility and if $y_1, \cdots y_k$ are linearly independent solutions of (5), (6), it is clear that the general solution is

$$c_1y_1 + c_2y_2 + \cdots + c_ky_k$$

where $c_1, \cdots c_k$ are arbitrary constants.

THEOREM 1. *If* $y_1, \cdots y_n$ *is a fundamental system of* (5), *a necessary and sufficient condition that the system* (5), (6) *be compatible is that the determinant*

$$D = \begin{vmatrix} W_1(y_1) & \cdots & W_1(y_n) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ W_n(y_1) & \cdots & W_n(y_n) \end{vmatrix}$$

*vanish.** *A necessary and sufficient condition that* (5), (6) *have k-fold compatibility is that D be of rank* $n - k$.

For if we substitute the general solution of (5)

$$u = c_1y_1 + \cdots + c_ny_n$$

into (6), we obtain a system of homogeneous linear equations for determining the c's whose determinant is D.

We shall naturally call the conditions (6) linearly dependent when and only when the n sets of $2n$ constants each

$$\alpha_i, \alpha_i', \cdots \alpha_i^{[n-1]}, \beta_i, \beta_i', \cdots \beta_i^{[n-1]} \qquad (i = 1, 2, \cdots n)$$

are linearly dependent. If this is the case, some of the conditions (6) are consequences of the rest, and the system (6) is equivalent to a system of the same form containing less than n equations. Conversely, a system of the form (6) but containing less than n equations is obviously equivalent to a linearly dependent system containing n equations. On the other hand, if the system (6) is linearly dependent, $D = 0$. Hence

THEOREM 2. *There always exists a solution of* (5) *not identically zero which satisfies fewer than n conditions of the form* (6).

* Cf. Birkhoff, loc. cit.

On the other hand, we readily see from Theorem 1 that if the conditions (6) are linearly independent, the system (5), (6) is *in general* incompatible, since in the space of $2n^2$ dimensions determined by the coordinates $\alpha_i, \cdots \alpha_i^{[n-1]}, \beta_i, \cdots \beta_i^{[n-1]}$ the equation $D = 0$ determines an algebraic manifold of only $2n^2 - 1$ dimensions.*

THEOREM 3. *A necessary and sufficient condition that the system* (1), (4) *have one and only one solution is that the reduced system* (5), (6) *be incompatible.*

A considerable part of this theorem follows at once from the obvious fact that *if* (1), (4) *admits of a solution at all, its general solution may be obtained by adding to any particular solution of* (1), (4) *the general solution of the reduced system* (5), (6). If then (5), (6) is compatible, (1), (4) has either no solution or an infinite number of solutions; while if (5), (6) is incompatible (1), (4) cannot have more than one solution. To prove our theorem completely, we need, then, merely to show that *if* (5), (6) *is incompatible,* (1), (4) *admits a solution.*

To prove this we begin with the semi-homogeneous case $p \equiv 0$. We see then from Theorem 2 that there exist n solutions $y_1, \cdots y_n$ of (1), no one of which vanishes identically, and such that y_i satisfies all of the reduced conditions (6) *except the ith.* Since the system (5), (6) is to be incompatible, none of the constants $W_i(y_i)$ are zero. Consequently the function

$$\frac{\gamma_1 y_1}{W_1(y_1)} + \cdots + \frac{\gamma_n y_n}{W_n(y_n)}$$

is a solution of (1) which satisfies all the conditions (4).

Turning now to the general case, let us denote by u any particular solution of (1), and by y the solution of (5) which satisfies the conditions

$$W_i(y) = \gamma_i - W_i(u) \qquad (i = 1, 2, \cdots n).$$

The existence of this function y is established by the part of the theorem we have just proved. The function $u + y$ is now clearly a solution of (1), (4). Thus our theorem is proved.

II. Green's Functions.

2. Definition and Condition for Existence.—We may arrive at the conception of the Green's Function for the system (5), (6) by attempting, in case the system is incompatible, to find a function not indentically zero

* To clinch this argument it is necessary to know that D does not vanish identically, i. e., for all possible conditions (6). This can be shown by exhibiting any particular system (6) for which $D \neq 0$; and such a system is that in which $\alpha_1 = \alpha_2' = \alpha_3'' = \ldots = \alpha_n^{[n-1]} = 1$ while all the other α's and all the β's are zero.

which comes as near as possible to being a solution of the system, the only failure lying in a discontinuity in its $(n-1)$th derivative at a single point ξ. Since it is only when such a function exists for *every* value of ξ between a and b that it proves to be of any importance, we lay down the definition as follows:

DEFINITION. *By a Green's Function $G(x, \xi)$ of the system* (5), (6), *where we assume the conditions* (6) *to be linearly independent, we understand a function of (x, ξ) defined when $a \leqq x \leqq b$, $a < \xi < b$, and which for every such value of ξ when regarded as a function of x alone has the following properties*:

1) *Throughout the interval $a \leqq x \leqq b$ it is continuous and has continuous derivatives of the first $n-2$ orders.*

2) *At every point of the interval $a \leqq x \leqq b$ except $x = \xi$ it satisfies* (5).

3) *It fulfills the boundary conditions* (6).

4) *At ξ it has a forward derivative of order $n-1$, D_+, and a backward derivative of order $n-1$, D_-; and*

$$D_+ - D_- = 1.$$

It will be noticed that in this definition we do not demand that the system (5), (6) be incompatible; and even if it is incompatible we do not as yet know that a Green's function will exist, or, if it does exist, whether it will be uniquely determined. We proceed to investigate these questions.

Let $y_1, \cdots y_n$ be a fundamental system of (5), and form with undetermined coefficients the two solutions

$$u_1(x) = c_1 y_1(x) + \cdots + c_n y_n(x),$$

$$u_2(x) = d_1 y_1(x) + \cdots + d_n y_n(x).$$

The most general function which satisfies 2) of our definition is

$$G(x, \xi) = \begin{cases} u_1(x) & a \leqq x \leqq \xi \\ u_2(x) & \xi \leqq x \leqq b. \end{cases} \tag{7}$$

In order that conditions 1) and 4) of our definition be satisfied by this function, it is necessary and sufficient that the c's and d's satisfy the following equations:

$$\begin{aligned} &d_1 y_1^{[i]}(\xi) + \cdots + d_n y_n^{[i]}(\xi) - c_1 y_1^{[i]}(\xi) - \cdots - c_n y_n^{[i]}(\xi) = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 0, 1, \cdots n-2) \\ &d_1 y_1^{[n-1]}(\xi) + \cdots + d_n y_n^{[n-1]}c(\xi) - c_1 y_1^{[n-1]}(\xi) - \cdots - c_n y_n^{[n-1]}(\xi) = 1. \end{aligned} \tag{8}$$

This may be regarded as a system of linear equations for determining the n differences $d_i - c_i$. The determinant of these equations, being the

Wronskian of the y's, is not zero since the y's are linearly independent. The equations (8) have therefore one and only one solution which we will denote by $z_1, \cdots z_n$. These quantities, like the c's and d's, are of course functions of ξ, and we write explicitly

$$d_i(\xi) - c_i(\xi) = z_i(\xi). \tag{9}$$

These functions z_i may be immediately written out each as the ratio of two determinants by Cramer's Rule. They are precisely the functions defined by Frobenius* as the functions *adjoint* to $y_1, \cdots y_n$. It is evident that these functions are continuous throughout the interval $a \leqq \xi \leqq b$. Apart from this fact we need for the moment only one property of $z_1, \cdots z_n$, namely that they are linearly independent.†

In order that the function G in (7) satisfy condition 3) of our definition, it is necessary and sufficient that the c's and d's satisfy the following equations:

$$c_1 A_i(y_1) + \cdots + c_n A_i(y_n) + d_1 B_i(y_1) + \cdots + d_n B_i(y_n) = 0 \qquad (i = 1, 2, \cdots n), \tag{10}$$

or, after the c's have been replaced by their values from (9)

$$d_1 W_i(y_1) + \cdots + d_n W_i(y_n) = z_1 A_i(y_1) + \cdots + z_n A_i(y_n) \qquad (i = 1, 2, \cdots n). \tag{11}$$

This is a system of linear equations for determining the d's concerning which we will establish the following

LEMMA. *A necessary and sufficient condition that equations* (11) *be consistent for all values of* ξ *such that* $a < \xi < b$ *is that their determinant* D *do not vanish.*

That this is a sufficient condition is obvious from Cramer's Rule. To prove it necessary we suppose $D = 0$, and have to show that equations (11) are inconsistent. From the vanishing of D we infer that there exist n constants $k_1, \cdots k_n$, not all zero, such that

$$k_1 W_1(y_i) + \cdots + k_n W_n(y_i) = 0 \qquad (i = 1, 2, \cdots n). \tag{12}$$

By multiplying equations (11) respectively by $k_1, \cdots k_n$ and adding, we could, if these equations were consistent, infer that

* Crelle, vol. 77 (1874), p. 250.

† The simple proof of this fact which consists in noticing that the Wronskian of the z's cannot vanish since the product of this Wronskian by the Wronskian of the y's is 1, is not available for us unless we make sufficient assumptions concerning the coefficients of (5) to secure the existence of the first $n-1$ derivatives of the z's. Cf., however, an article by the author: Bull. Amer. Math. Soc., 2d Ser., vol. 8 (1901), p. 53. See particularly Theorem 14, p. 61.

$$0 = z_1 \sum_{j=1}^{n} k_j A_j(y_1) + \cdots + z_n \sum_{j=1}^{n} k_j A_j(y_n).$$

Since this is an identity in ξ, and the z's are linearly independent, it follows that

$$(13) \qquad k_1 A_1(y_i) + \cdots + k_n A_n(y_i) = 0 \qquad (i = 1, 2, \cdots n).$$

Subtracting these equations from (12), we find

$$(14) \qquad k_1 B_1(y_i) + \cdots + k_n B_n(y_i) = 0 \qquad (i = 1, 2, \cdots n).$$

If we use formula (2) for the A's and B's, (13) and (14) take the form

$$(15) \quad (k_1\alpha_1 + \cdots + k_n\alpha_n)y_i(a) + \cdots + (k_1\alpha_1^{[n-1]} + \cdots + k_n\alpha_n^{[n-1]})y_i^{[n-1]}(a) = 0$$

$$(16) \quad (k_1\beta_1 + \cdots + k_n\beta_n)y_i(b) + \cdots + (k_1\beta_1^{[n-1]} + \cdots + k_n\beta_n^{[n-1]})y_i^{[n-1]}(b) = 0$$

$$(i = 1, 2, \cdots n).$$

We may regard (15) as a sytem of linear equations for determining the n quantities which occur in it in parentheses. The determinant of this system is the Wronskian of $y_1, \cdots y_n$ taken at the point $x = a$, which, since the y's are linearly independent, is not zero. Consequently

$$(17) \qquad k_1\alpha_1^{[j]} + \cdots + k_n\alpha_n^{[j]} = 0 \qquad (j = 0, 1, \cdots n - 1).$$

Similarly we infer from (16) that

$$(18) \qquad k_1\beta_1^{[j]} + \cdots + k_n\beta_n^{[j]} = 0 \qquad (j = 0, 1, \cdots n - 1).$$

These relations (17), (18) mean, however, that the conditions (6) are linearly dependent, a possibility, we have explicitly ruled out in the hypothesis of our definition of a Green's Function. One lemma is thus proved, since the assumption that it is false has led to a contradiction.

If we now notice that the D in our lemma is the same as the D in Theorem 1, we have the result:

THEOREM 4. *A necessary and sufficient condition for the existence of a Green's Function of the system* (5), (6), *where the boundary conditions* (6) *are assumed to be linearly independent, is that the system be incompatible.*

Since, moreover, if this condition is fulfilled, the determinant D of equations (11) is not zero, only one determination of the d's, and therefore by (9) of the c's, is possible. Hence

COROLLARY. *If the system* (5), (6) *is incompatible, it has only one Green's Function.*

The Green's Function has been defined so far only when ξ is equal neither to a nor to b. We now lay down the further

Definition. *The formula* (7) *which gives the Green's Function when* $a < \xi < b$ *shall serve to define it when* $\xi = a$ *and when* $\xi = b$.

Since the c's and d's obtained from (11) and (9) are clearly continuous functions of ξ when $a \leqq \xi \leqq b$, and $y_1, \cdots y_n$ are continuous together with their first $n-2$ derivatives when $a \leqq x \leqq b$, it follows from (7) that $G(x, \xi)$ and its first $n-2$ derivatives with regard to x are continuous in each of the triangles $a \leqq x \leqq \xi \leqq b$ and $a \leqq \xi \leqq x \leqq b$. Moreover none of these functions has any discontinuity on the line $x = \xi$. Hence

Theorem 5. *If the Green's Function of the system* (5), (6) *exists, it is a continuous function of* (x, ξ) *when* $a \leqq x \leqq b$, $a \leqq \xi \leqq b$, *and the same is true of its first* $n-2$ *derivatives with regard to* x.

3. The Fundamental Application of Green's Functions.—The general solution of the problem (1), (4) is clearly the sum of the general solutions of the semi-homogeneous problems (1), (6) and (5), (4). We have had occasion to consider the latter problem incidentally in §1. If the reduced system (5), (6) is incompatible, the former problem has, as we know, one and only one solution. We wish to prove in this section that this solution is given by the formula

$$u(x) = \int_a^b G(x, \xi) p(\xi) d\xi, \tag{19}$$

where G is the Green's Function of (5), (6). Referring to formula (7), we see that we can write

$$u(x) = \int_a^x [d_1(\xi) y_1(x) + \cdots + d_n(\xi) y_n(x)] p(\xi) d\xi$$
$$- \int_b^x [c_1(\xi) y_1(x) + \cdots + c_n(\xi) y_n(x)] p(\xi) d\xi.$$

If we differentiate and simplify by means of (8), we find:

$$u^{[i]}(x) = \int_a^x [d_1(\xi) y_1^{[i]}(x) + \cdots + d_n(\xi) y_n^{[i]}(x)] p(\xi) d\xi$$
$$- \int_b^x [c_1(\xi) y_1^{[i]}(x) + \cdots + c_n(\xi) y_n^{[i]}(x)] p(\xi) d\xi \qquad (i = 1, \cdots n-1),$$
$$u^{[n]}(x) = \int_a^x [d_1(\xi) y_1^{[n]}(x) + \cdots + d_n(\xi) y_n^{[n]}(x)] p(\xi) d\xi$$
$$- \int_b^x [c_1(\xi) y_1^{[n]}(x) + \cdots + c_n(\xi) y_n^{[n]}(x)] p(\xi) d\xi + p(x).$$

Substituting these values of u and its derivatives in (1), and remembering that the y's are solutions of (5), we see that the function u determined by (19) satisfies (1). We also find from the values of u and its derivatives just obtained that

$$\left.\begin{aligned} A_i(u) &= \int_a^b [c_1(\xi)A_i(y_1) + \cdots + c_n(\xi)A_i(y_n)]p(\xi)d\xi \\ B_i(u) &= \int_a^b [d_1(\xi)B_i(y_1) + \cdots + d_n(\xi)B_i(y_n)]p(\xi)d\xi \end{aligned}\right\} \qquad (i = 1, 2, \cdots n).$$

A reference to (10) shows that the sum of these two quantities is zero. Thus we have proved

THEOREM 6. *If the system* (5), (6) *has a Green's Function* $G(x, \xi)$, *the solution of the semi-homogeneous problem* (1), (6) *is given by* (19), *and the first* $n - 1$ *derivatives of* u *may be obtained by differentiating* (19) *under the integral sign.*

In the special case in which $\alpha_1 = \alpha_2' = \cdots = \alpha_n^{[n-1]} = 1$, while all the other α's and all the β's are zero, formula (19) gives the *principal solution* of (1) at the point a, that is the solution which with its first $n - 1$ derivatives vanishes at a. In this case the c's are all zero by (10), and hence, by (9), the d's are the functions adjoint to the y's. Hence formula (19) reduces to the standard formula (Schlesinger's Handbuch, vol. 1, p. 78, (6)), obtainable by the method of variation of constants or otherwise, for this principal solution.

4. **Further Properties of Green's Functions.**—The facts to which we now come are less fundamental than those we have so far considered but they are commonly given great prominence in the development of the theory, and they are in themselves not without interest. In order to obtain them we must impose additional restrictions on the coefficients of (5).

We assume that $p_i(x)$ *has continuous derivatives of the first* $n - i$ *orders.*

This is sufficient to insure the existence and continuity of the first n derivatives of the adjoint functions $z_1(\xi), \cdots z_n(\xi)$, and to cause them to satisfy the adjoint differential equation:*

$$(20) \qquad (-1)^n \frac{d^n z}{d\xi^n} + (-1)^{n-1} \frac{d^{n-1}(p_1 z)}{d\xi^{n-1}} + \cdots - \frac{d(p_{n-1}z)}{d\xi} + p_n z = 0.$$

From (11) and (9) we see that the c's and d's are linear homogeneous combinations with constant coefficients of the z's. Consequently the c's and d's are also solutions of (20). Turning to (7), we see then that when x is constant, G also satisfies (20) for all values of ξ in the interval $a \leqq \xi \leqq b$

* This may be seen as follows:

Even with the mere assumption of continuity on the functions p_i, it is known (cf. Bull. Amer. Math. Soc., 2d Ser., vol. 8 (1901), p. 63) that the multipliers (integrating factors) of (5) constitute a linear family of which $z_1, \ldots z_n$ forms a basis. With the additional restrictions concerning the derivatives of the p's, it is known from Lagrange's Identity that all solutions of (20) are multipliers of (5). These solutions form a linear family involving n linearly independent functions. Consequently the solutions of (20) are identical with the linear family $c_1z_1 + \ldots + c_nz_n$.

except $\xi = x$, and that even at this point it satisfies (20) if we agree to consider only forward or only backward derivatives.

In order to compare these forward and backward derivatives, that is the derivatives with regard to ξ of u_1 and u_2 at the point $\xi = x$, we turn to equations (8). Differentiate each of these equations with regard to ξ, remembering that the c's and d's are themselves functions of ξ, and subtract from each equation thus obtained the next following equation (8). We thus get the system of $n - 1$ equations:

$$
(8')\quad
\begin{aligned}
&d_1' y_1^{[i]}(\xi) + \cdots + d_n' y_n^{[i]}(\xi) - c_1' y_1^{[i]}(\xi) - \cdots - c_n' y_n^{[i]}(\xi) = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i=0,\ 1,\ \cdots\ n-3), \\
&d_1' y_1^{[n-2]}(\xi) + \cdots + d_n' y_n^{[n-2]}(\xi) - c_1' y_1^{[n-2]}(\xi) - \cdots - c_n' y_n^{[n-2]}(\xi) = -1.
\end{aligned}
$$

If here we differentiate each equation and subtract from it the next following equation (8′), we get a system of $n - 2$ equations (8″), etc. Finally we collect together the *first* equations of the sets (8), (8′), (8″), $\cdots$, and thus obtain a new system

$$
(21)\quad
\begin{aligned}
&d_1^{[i]}(\xi) y_1(\xi) + \cdots + d_n^{[i]}(\xi) y_n(\xi) - c_1^{[i]}(\xi) y_1(\xi) - \cdots - c_n^{[i]}(\xi) y_n(\xi) = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 0,\ 1,\ \cdots\ n-2), \\
&d_1^{[n-1]}(\xi) y_1(\xi) + \cdots + d_n^{[n-1]}(\xi) y_n(\xi) \\
&\qquad - c_1^{[n-1]}(\xi) y_1(\xi) - \cdots - c_n^{[n-1]}(\xi) y_n(\xi) = (-1)^{n-1}.
\end{aligned}
$$

These equations show us that the first $n - 2$ forward derivatives of G with regard to ξ at the point $\xi = x$ have the same values respectively as the corresponding backward derivatives; while if we denote by Δ_+ and Δ_- the $(n - 1)$th forward and backward derivatives respectively at $\xi = x$,

$$\Delta_+ - \Delta_- = (-1)^n.$$

We thus see that, except for the boundary conditions, G when regarded as a function of ξ, x being a parameter satisfies precisely the conditions for the Green's function of equation (20), or if n is odd, for the negative of this Green's function. Let us then see whether we can find a system of boundary conditions of the form (6) satisfied by G for each (constant) value of x. The important point here is that the coefficients α, β must be independent of of x.

Let us write

$$
\begin{aligned}
A(\varphi) &= \alpha\varphi(a) + \cdots + \alpha^{[n-1]}\varphi^{[n-1]}(a), \\
B(\varphi) &= \beta\varphi(b) + \cdots + \beta^{[n-1]}\varphi^{[n-1]}(b), \\
W(\varphi) &= A(\varphi) + B(\varphi),
\end{aligned}
$$

where the α's and β's are undetermined constants. We wish to determine these constants in the most general way possible so that $G(x, \xi)$ regarded as a function of ξ shall satisfy the relation $W(G) = 0$. From (7) we have (x being regarded as a constant in u_1, u_2 as well as in G):

$$W(G) = A(u_2) + B(u_1) = y_1(x)[A(d_1) + B(c_1)] + \cdots + y_n(x)[A(d_n) + B(c_n)].$$

Consequently $W(G)$ vanishes for all values of x when and only when all the coefficients of $y_1, \cdots y_n$ in this expression are zero; that is when and only when the α's and β's satisfy the following equations:

$$(22)\quad \alpha d_i(a) + \cdots + \alpha^{[n-1]} d_i^{[n-1]}(a) + \beta c_i(b) + \cdots + \beta^{[n-1]} c_i^{[n-1]}(b) = 0 \qquad (i = 1, 2, \cdots n).$$

LEMMA. *The equations* (22) *are linearly independent.*

Suppose this were not the case. Then there would exist n constants $k_1, \cdots k_n$ not all zero and such that

$$\left.\begin{aligned} k_1 d_1^{[j]}(a) + k_2 d_2^{[j]}(a) + \cdots + k_n d_n^{[j]}(a) &= 0 \\ k_1 c_1^{[j]}(a) + k_2 c_2^{[j]}(a) + \cdots + k_n c_n^{[j]}(a) &= 0. \end{aligned}\right. \qquad (j = 0, 1, \cdots n - 1).$$

Consequently the functions

$$k_1 d_1(\xi) + \cdots + k_n d_n(\xi),$$
$$k_1 c_1(\xi) + \cdots + k_n c_n(\xi)$$

vanish, together with their first $n - 1$ derivatives, at the points a and b respectively; so that, being solutions of (20), they vanish identically. Accordingly their difference, which by (9) is

$$k_1 z_1(\xi) + \cdots + k_n z_n(\xi),$$

vanishes identically. But this is impossible since, as we noted in §2, $z_1, \cdots z_n$ are linearly independent.

The equations (22), being thus linearly independent, have n linearly independent solutions, which we denote by

$$\bar{\alpha}_i, \bar{\alpha}_i', \cdots \bar{\alpha}_i^{[n-1]}, \bar{\beta}_i, \bar{\beta}_i', \cdots \bar{\beta}_i^{[n-1]} \qquad (i = 1, 2, \cdots n),$$

upon which all other solutions are linearly dependent.

DEFINITION. *The linearly independent boundary conditions*

$$(23)\quad \bar{\alpha}_i u(a) + \cdots + \bar{\alpha}_i^{[n-1]} u^{[n-1]}(a) + \bar{\beta}_i u(b) + \cdots + \bar{\beta}_i^{[n-1]} u^{[n-1]}(b) = 0 \qquad (i = 1, 2, \cdots n),$$

together with the equation (20) *form a homogeneous system which we call the*

system adjoint to the system (5), (6). *In forming this system it is assumed that the system* (5), (6) *is incompatible.**

The facts proved in this section may now be summed up in the following theorem:

THEOREM 7. *If the system* (5), (6) *has a Green's function* $G(x, \xi)$, *then* $(-1)^n G(x, \xi)$ *is the Green's function of the adjoint system* (20), (23).†

From this it follows that the system (20), (23) is also incompatible since it is only incompatible systems which have a Green's function. Moreover the system adjoint to (20), (23) has, by Theorem 7, $G(x, \xi)$ as its Green's function. The differential equation of this system is (5), since any homogeneous linear differential equation is the adjoint of its adjoint. Moreover the boundary conditions of the system adjoint to (20), (23) are easily seen to be conditions (6), since in the proof of Theorem 7 it was shown that there is essentially only one system of boundary conditions in the parameter which a Green's function satisfies. Hence

THEOREM 8. *If the system* (5), (6) *is incompatible, its adjoint is also incompatible, and the adjoint of this last named system is the original system* (5), (6).

The theory of Green's functions can be put into a more elegant form by considering, with Hilbert, not the differential *equation* (5) but the differential *expression*

$$L(u) \equiv a_0 \frac{d^n u}{dx^n} + a_1 \frac{d^{n-1} u}{dx^{n-1}} + \cdots + a_n u, \tag{23}$$

where the a's are functions of x continuous throughout the interval $a \leqq x \leqq b$, a_i has continuous derivatives of the first $n - i$ orders, and a_0 does not vanish at any point of this interval.

DEFINITION. *By the Green's function of the system consisting of the expression* (23) *and the conditions* (6) *is understood the quotient of the Green's function of the system consisting of the equation* $L(u) = 0$ *and the conditions* (6) *by* $a_0(\xi)$.

The properties of Green's functions in this sense follow at once from the results already obtained. For instance, if we denote the Green's function in the new sense by $G(x, \xi)$, it is still true that the solution of the non-homogeneous equation

$$L(u) = p$$

* The term *adjoint* in this sense and the conception with this degree of generality was first used by Birkhoff, loc. cit. Indeed he defines the conception in a more general case since he does not restrict the system (5), (6) to be incompatible. Cf. also Bounitzki, loc. cit. In a special case the conception occurs in a paper of Liouville (Liouville's Journal, vol. 3 (1838), p. 561). That this conception plays an important part in the theory of Green's functions was clearly indicated by the present writer in 1901 (cf. loc. cit.).

† The factor $(-1)^n$ was omitted by an oversight in my paper of 1901.

which satisfies the homogeneous conditions (6) is, provided G exists, given by formula (19).

Again, the system adjoint to (23), (6) consists of the expression adjoint to $L(u)$ and of the boundary conditions which $G(x, \xi)$ satisfies when regarded as a function of ξ. These boundary conditions, however, are not the same as those satisfied by the Green's function of the *equation* $L(u) = 0$ together with (6). With this understanding we may say that the Green's function of (23), (6) is, when regarded as a function of ξ, precisely the Green's function of the adjoint system. It seems hardly necessary to multiply illustrations or to go into proofs here.

We close by noting that *the Green's function of the expresssion* (23) *together with the conditions* (6) *is a covariant both with regard to change of independent and of dependent variable of the system consisting of the equation* $L(u) = 0$ *together with the conditions* (6). That is, if a change of independent variable

$$t = f(x), \quad \tau = f(\xi),$$

where $f(x)$ has continuous derivatives of the first n orders, and $f'(x)$ does not vanish in the interval $a \leqq x \leqq b$, is made both in (23) and in (6), the Green's function of the transformed system is equal to the Green's function of the original system multiplied by a power (the (-1)th) of $f'(\xi)$; and on the other hand a change of independent variable

$$u = \psi \cdot \eta,$$

where ψ is a given non-vanishing function of x which in the interval $a \leqq x \leqq b$ is continuous and has continuous derivatives of the first n orders, carries over the system (23), (6) into a system whose Green's function is equal to the product of the original Green's function by a power (the (-1)th) of $\psi(x)$ And finally, if the expression (23) is multiplied by a non-vanishing function $\varphi(x)$, which together with its first n derivatives is continuous in the interval $a \leqq x \leqq b$, the Green's function of the new system is equal to the original Green's function multiplied by a power (the (-1)th) of $\varphi(\xi)$.

The truth of these statements follows readily from the definition of Green's functions. As in § 2, we need not demand the existence of derivatives of the coefficients a_i.

III. The Linear Difference Equation.

5. Linear Boundary Problems.—We consider here the difference equation

$$(24) \quad p_0(x)u(x+n) + p_1(x)u(x+n-1) + \cdots + p_n(x)u(x) = p(x)$$
$$(x = a, a+1, \cdots b-n),$$

where a and b, and consequently x, may without loss of generality be regarded as integers. We will assume that $p_0(x)$ and $p_n(x)$ do not vanish for any value of x from a to $b-n$ inclusive; and that $b - a \geqq n$.

Along with (24) we consider certain boundary conditions which we denote as follows:

If $\varphi(x)$ is a function of the integer x defined when $a \leqq x \leqq b$, we write

$$(25)\quad \begin{aligned} A_i(\varphi) &= \alpha_{n-1}^{[i]}\varphi(a+n-1) + \alpha_{n-2}^{[i]}\varphi(a+n-2) + \cdots + \alpha_0^{[i]}\varphi(a) \\ B_i(\varphi) &= \beta_{n-1}^{[i]}\varphi(b-n+1) + \beta_{n-2}^{[i]}\varphi(b-n+2) + \cdots + \beta_0^{[i]}\varphi(b) \end{aligned} \qquad (i = 1, 2, \cdots n),$$

where the α's and β's are constants, i. e., independent of x. If, now, we let

$$(26)\qquad W_i(\varphi) = A_i(\varphi) + B_i(\varphi),$$

we may write our boundary conditions as follows, the γ's being constants:

$$(27)\qquad W_i(u) = \gamma_i \qquad (i = 1, 2, \cdots n)$$

The system (24), (27) is in general non-homogeneous, and we distinguish between semi-homogeneous and homogeneous systems precisely as in the analogous case of §1. The *reduced system* is

$$(28)\qquad p_0(x)u(x+n) + p_1(x)u(x+n-1) + \cdots + p_n(x)u(x) = 0 \qquad (x = a, a+1, \cdots b-n),$$

$$(29)\qquad W_i(u) = 0 \qquad (i = 1, 2, \cdots n).$$

We define compatibility, k-fold compatibility, for the homogeneous system (28), (29) precisely as in §1, and we see at once that Theorems 1, 2 hold here also if (5), (6) be replaced by (28), (29). We also see, as in §1, that if the conditions (29) are linearly independent, the system (28), (29) is *in general* incompatible, the special conditions (29) which show that the determinant D does not always vanish being

$$u(a) = u(a+1) = \cdots = u(a+n-1) = 0.$$

Finally the fundamental theorem here is

THEOREM 9. *A necessary and sufficient condition that the system* (24), (27) *have one and only one solution is that the reduced system* (28), (29) *be incompatible.*

The proof of this theorem is identical with the proof of Theorem 3 in §1.

6. Green's Functions for Linear Difference Equations.—Here too the conception of a Green's Function may be regarded as having its rise in the attempt to form a function not identically zero which comes as near as possible to satisfying the system (28), (29) when this system is incompatible.

DEFINITION. *By a Green's function $G(x, \xi)$ of the system consisting of the expression in* (28), *and conditions* (29), *where we assume that the conditions* (29) *are linearly independent, we understand a function of the integral arguments (x, ξ) defined when $a \leqq x \leqq b$, $a \leqq \xi \leqq b - n$, and such that when ξ is fixed, G regarded as a function of x satisfies the boundary conditions* (29); *and, except for the one value $x = \xi$, the equation* (28); *and is such that when substituted in the first member of* (28), *this first member takes the value* 1 *when $x = \xi$.*

In order to see under what circumstances such a function exists, we consider a system $y_1, \cdots y_n$ of linearly independent solutions of (28), and form from them with undetermined coefficients the further solutions

$$(30)\qquad \begin{aligned} u_1(x) &= c_1 y_1(x) + \cdots + c_n y_n(x), \\ u_2(x) &= d_1 y_1(x) + \cdots + d_n y_n(x), \end{aligned}$$

where the c's and d's are ultimately to be functions of ξ; a fact which will be indicated when necessary.

The most general function which satisfies (28) except when x has one of the values $\xi, \xi - 1, \cdots \xi - n + 1$ lying in the interval $a \leqq x \leqq b - n$ is then

$$(31)\qquad u(x) = \begin{cases} u_1(x) & x = a, a + 1, \cdots \xi, \\ u_2(x) & x = \xi + 1, \xi + 2, \cdots b. \end{cases}$$

In order that $u(x)$ satisfy (28) also at such of the points $x = \xi - 1, \cdots \xi - n + 1$ as satisfy the inequality $a \leqq x$ it is evidently necessary* and sufficient that

$$(32)\qquad u_2(x) = u_1(x) \qquad x = \xi + 1, \xi + 2, \cdots \xi + n - 1.$$

In order that the condition demanded in the last clause of our definition be fulfilled it is then necessary and sufficient that

$$(33)\qquad p_0(\xi)[u_2(\xi + n) - u_1(\xi + n)] = 1.$$

Substituting in (32), (33) the value of u_1 and u_2 from (30), and setting

$$(34)\qquad z_i(\xi) = p_0(\xi)(d_i(\xi) - c_i(\xi)),$$

we find as an equivalent form for conditions (32), (33)

$$(35)\qquad \begin{aligned} z_1(\xi)y_1(\xi + i) + \cdots + z_n(\xi)y_n(\xi + i) &= 0 \qquad (i = 1, 2, \cdots n-1) \\ z_1(\xi)y_1(\xi + n) + \cdots + z_n(\xi)y_n(\xi + n) &= 1. \end{aligned}$$

* Strictly speaking this is necessary only when $a \leqq \xi - n + 1$, so that *all* the points in question lie in the interval ab. If some of the points $\xi - 1, \ldots \xi - n + 1$ lie outside of the interval ab, some of the conditions (32) need not be imposed. They may, however, in this case be imposed *without at all affecting the function $u(x)$*.

This is a system of linear equations for determining the z's, whose determinant, since the y's are linearly independent solutions of (28), is not zero. The functions z are therefore uniquely determined and are called the functions *adjoint* to $y_1, \cdots y_n$. It should be noticed that whereas the y's are defined when $a \leqq x \leqq b$, the z's are defined only when $a \leqq \xi \leqq b - n$.

LEMMA. *The functions $z_1(\xi), \cdots z_n(\xi)$ are linearly independent.*

If they were not, the determinant

$$\Delta = \begin{vmatrix} z_1(\xi) & \cdots & z_n(\xi) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ z_1(\xi - n + 1) & \cdots & z_n(\xi - n + 1) \end{vmatrix}$$

would necessarily vanish for all values of ξ from a to $b - n$ inclusive. This is impossible since, when we form the product of Δ by the determinant

$$\begin{vmatrix} y_1(\xi + 1) & \cdots & y_n(\xi + 1) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ y_1(\xi + n) & \cdots & y_n(\xi + n) \end{vmatrix}$$

by combining rows with rows, we get, as we see from (35), a determinant which has zeros everywhere above the secondary diagonal and ones at every point in this diagonal, and which is therefore not zero. This proves our lemma.

Returning now to the function $u(x)$ defined by (31), let us see whether we can make it satisfy the boundary conditions (29). For this purpose it is necessary and sufficient that the quantities c, d satisfy the relations

$$c_1 A_i(y_1) + \cdots + c_n A_i(y_n) + d_1 B_i(y_1) + \cdots + d_n B_i(y_n) = 0 \qquad (i = 1, 2, \cdots n). \tag{36}$$

Substituting here for the c's their values from (34), we find

$$\begin{aligned} &p_0(\xi)[d_1(\xi) W_i(y_1) + \cdots + d_n(\xi) W_i(y_n)] \\ &\qquad = z_1(\xi) A_i(y_1) + \cdots + z_n(\xi) A_i(y_n) \quad (i = 1, 2, \cdots n). \end{aligned} \tag{37}$$

LEMMA. *A necessary and sufficient condition that equations* (37) *be consistent for all values of ξ for which $a \leqq \xi \leqq b - n$ is that the determinant D of Theorem 1 do not vanish.*

The proof of this lemma follows so closely the proof of the lemma preceding Theorem 4 that it is unnecessary to repeat it.

We now obtain at once

THEOREM 10. *A necessary and sufficient condition for the existence of a Green's function of the system, consisting of the expression in* (28) *together with conditions* (29), *where* (29) *are assumed to be linearly independent, is that the system* (28), (29) *be incompatible.*

COROLLARY. *If the system* (28), (29) *is incompatible, there exists only one Green's function.*

7. An Application of Green's Function.—If the Green's function G used in §6 exists, we know that the semi-homogeneous system (24), (29) has one and only one solution. We wish to prove in this section that this solution is given by the formula

$$u(x) = \sum_{\xi=a}^{b-n} G(x, \xi)p(\xi). \tag{38}$$

The function u determined by (38) may be written

$$\begin{aligned} u(x) = &\sum_{\xi=a}^{x-1} [d_1(\xi)y_1(x) + \cdots + d_n(\xi)y_n(x)]p(\xi) \\ &+ \sum_{\xi=x}^{b-n} [c_1(\xi)y_1(x) + \cdots + c_n(\xi)y_n(x)]p(\xi). \end{aligned} \tag{39}$$

Using formulæ (35), and assuming $x \leqq b - n + 1$, we see that we may write:

$$\begin{aligned} u(x + i) = &\sum_{\xi=a}^{x-1} [d_1(\xi)y_1(x + i) + \cdots + d_n(\xi)y_n(x + i)]p(\xi) \\ &+ \sum_{\xi=x}^{b-n} [c_1(\xi)y_1(x + i) + \cdots + c_n(\xi)y_n(x + i)]p(\xi) \quad (i = 1, 2, \cdots n - 1). \end{aligned}$$

Finally if $x \leqq b - n$

$$\begin{aligned} u(x + n) = &\sum_{\xi=a}^{x-1} [d_1(\xi)y_1(x + n) + \cdots + d_n(\xi)y_n(x + n)]p(\xi) \\ &+ \sum_{\xi=x}^{b-n} [c_1(\xi)y_1(x + n) + \cdots + c_n(\xi)y_n(x + n)]p(\xi) + \frac{p(x)}{p_0(x)}. \end{aligned}$$

From these equations it follows that, since the y's are solutions of (28), $u(x)$ satisfies (24). We also find from the values of $u(x)$, $u(x + 1)$, $\cdots u(x + n - 1)$ just obtained

$$A_i(u) = \sum_{\xi=a}^{b-n} [c_1(\xi)A_i(y_1) + \cdots + c_n(\xi)A_i(y_n)]p(\xi),$$

$$B_i(u) = \sum_{\xi=a}^{b-n} [d_1(\xi)B_i(y_1) + \cdots + d_n(\xi)B_i(y_n)]p(\xi).$$

A reference to (36) shows that the sum of these two quantities is zero; hence

THEOREM 11. *If the system consisting of the first member of* (24) *and the conditions* (29) *has a Green's function* $G(x, \xi)$, *the solution of the semi-homogeneous system* (24), (29) *is given by* (38).

Here too we may obtain, as a very special case, the formula for expressing the solution of (24) which vanishes at the points $a, a+1, \cdots a+n-1$ in terms of a fundamental system $y_1, \cdots y_n$ of the reduced equation (28). Here all the c's vanish identically by (36), while the d's are then determined by (34) in terms of the z's. Hence, from (39),

$$u(x) = \sum_{\xi=a}^{x-1} [z_1(\xi)y_1(x) + \cdots + z_n(\xi)y_n(x)] \frac{p(\xi)}{p_0(\xi)}$$

is the desired solution of (24),—the *principal solution* at the point a.

It would now be easy to proceed to parallel the developments of §4 by introducing the adjoint difference equation.* Enough has, however, already been said to indicate how completely the whole theory can be carried over from differential to difference equations.

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* Cf. for instance Wallenberg: Sitzungsberichte d. Berl. math. Ges., 7th year (1908), p. 50.

CONJUGATE DIRECTIONS ON A HYPERSURFACE IN A SPACE OF FOUR DIMENSIONS AND SOME ALLIED CURVES.

By C. L. E. Moore.

Introduction.—On a surface in a space of three dimensions the line joining two infinitely near points (x_1, x_2, x_3, x_4), $(x_1 + dx_1, x_2 + dx_2 \cdots)$, or as we shall say for brevity the points x and $x + dx$, and the line of intersection of the planes tangent to the surface at these two points define what are called conjugate directions. The correspondence between these lines is projective and the coincident or self-corresponding elements define the principal or asymptotic directions. This is one of the most important topics in differential geometry.

The idea of conjugate directions has been somewhat generalized by Segre.* He studied the correspondence between the planes determined by the fixed point x and the two neighboring points $x + dx$, $x + 2dx + d^2x$ (dx and d^2x are variable) and the point of intersection of the planes tangent to the surface at these points.

It is the object of the present paper to study the two corresponding problems for hypersurfaces in space of four dimensions. In each case we have a line-plane correspondence, which possesses properties similar to those for space of three dimensions. The correspondence between the planes determined by the fixed point x and the points $x + dx$, $x+2dx+d^2x$ and the line of intersection of the plane spaces tangent to the hypersurface at these three points forms the extension of Segre's problem. This correspondence is (1, 1). The correspondence between the plane S_3 determined by four infinitely near points and the point of intersection of the tangent spaces to the hypersurface at these points is (∞, ∞) and therefore is not discussed. Here four infinitely near points uniquely determine a single point of intersection of the tangent spaces but the same S_3 is determined by any four infinitely near points lying on the surface of intersection of the hypersurface with this S_3.

1. **Conjugate Directions.**—The tangent lines at a point x to a hypersurface in a space of four dimensions S_4 will generate a tangent hyperplane π_3. Two successive tangent hyperplanes will intersect in a plane which is tangent to the hypersurface at x, that is, the plane will contain a pencil of

* Complementi alla theoria delle tangenti coniugati di una superficie. Rendiconti dei Lincei, vol. XVII, page 405.

tangent lines passing through x. The common plane and the line joining the points of contact of two successive tangent hyperplanes are said to be *conjugate*. If we consider the hypersurface as the envelope of its tangent hyperplanes, the tangent π_3 at the point x and the tangent π_3 at the infinitely near point $x + dx$ can be taken as the hyperplanes ξ and $\xi + \delta\xi$ where the equation of the hyperplane is $\Sigma\xi x = 0$. If we consider a direction as defined by two infinitely near elements, whether points or hyperplanes, the two hyperplanes ξ and $\xi + \delta\xi$ will define a direction among the ∞^3 hyperplanes tangent to the hypersurface V_3. Then we will say that the directions x, $x + dx$ and ξ, $\xi + \delta\xi$ are conjugate. This agrees with the general conception of conjugate directions on a surface in an ordinary space S_3.

We saw that the common plane to two consecutive hyperplanes contains a pencil of lines (directions) through the point O. There we can look upon the line joining the points of contact as conjugate to this whole pencil of directions. This correspondence between the tangent lines and the tangent planes is a $(1, 1)$ correspondence. In fact, it is a polar reciprocation as can be shown as follows.

Let the equations of the hypersurface V_3 be

$$(F) \qquad x_i = x_i(u_1, u_2, u_3) \qquad (i = 1, 2, 3, 4, 5).$$

Now if we consider the u's as functions of a single variable t, x_i will describe a curve on V_3 passing through the point O whose coordinates are $(x_1, x_2, x_3, x_4, x_5)$ say. The tangent line to this curve will join x to $x + dx$. On the other hand if we consider V_3 as the envelope of its tangent hyperplanes then its equations are

$$(E) \qquad \xi_i = \xi_i(u_1, u_2, u_3) \qquad (i = 1, 2, 3, 4, 5).$$

In these equations (F) and (E) if we give any definite set of values to the u's the equation (F) will give the point of contact of the hyperplane (E). If the u's in (E) are the same functions of t as above, the hyperplanes (E) will be tangent to V_3 in the points of the above curve. Two successive tangent hyperplanes will intersect in a tangent plane and the correspondence sought is that between these planes and the tangent lines to the curve at the point of contact of the plane.

Consider the two hyperplanes obtained by giving to the parameters the values u and $u + \delta u$. These same values of the parameters put in equations (F) will give the points of contact of the two hyperplanes. Then on V_3 the direction determined by δu_i is the direction of the line joining the points of contact of the two hyperplanes. The equations of the two hyperplanes are

$$(1) \qquad (\xi, x) = 0, \quad (\xi + \delta\xi, x) = 0,$$

where

$$(\xi, x) = \Sigma\xi_i x_i = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 + \xi_5 x_5.$$

The second of equations (1) can be replaced, in consequence of the first, by

$$(\delta\xi, x) = 0.$$

The condition that this hyperplane should contain the point $x + dx$, since it already contains the point x is

$$(\delta\xi, dx) = 0.$$

The directions du are then the directions conjugate to the directions δu. All the directions du which are conjugate to δu form the plane of intersection of the two hyperplanes ξ and $\xi + \delta\xi$. The above relation may be put in the form

$$\left[\left(\frac{\partial\xi}{\partial u}, \delta u\right), \left(\frac{\partial x}{\partial u}, du\right)\right] = 0.$$

or

$$\Sigma\left(\frac{\partial\xi}{\partial u_i}, \frac{\partial x}{\partial u_j}\right)\delta u_i du_j = 0. \tag{2}$$

The tangent hyperplane is determined by the points

$$(\xi, x) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_1}\right) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_2}\right) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_3}\right) = 0 \tag{3}$$

(here the points are represented as the envelope of the hyperplanes which pass through them). Differentiating equations (3) except the first and remembering that both the ξ's and the x's are functions of the u's, we have

$$\left(\frac{\partial\xi}{\partial u_i}, \frac{\partial x}{\partial u_j}\right) + \left(\xi, \frac{\partial^2 x}{\partial u_i \partial u_j}\right) = 0 \qquad (i, j = 1, 2, 3).$$

Substituting the values of $\left(\frac{\partial\xi}{\partial u_i}, \frac{\partial x}{\partial u_j}\right)$ from these equations in (2), we have

$$\sum_{i,j}\left(\xi, \frac{\partial^2 x}{\partial u_i du_j}\right)\delta u_i du_j = 0,$$

or in still simpler form

$$\left(\xi, \Sigma\frac{\partial^2 x}{\partial u_i \partial u_j}\delta u_i du_j\right) = 0. \tag{4}$$

Eliminating ξ between (3) and (4) we have

$$
(5) \qquad \begin{vmatrix} x_1 & \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} & \Sigma \frac{\partial^2 x_1}{\partial u_i \partial u_j} \delta u_i du_j \\ x_2 & \frac{\partial x_2}{\partial u_1} & \cdot & \cdot & \cdot \\ x_3 & \frac{\partial x_3}{\partial u_1} & \cdot & \cdot & \cdot \\ x_4 & \frac{\partial x_4}{\partial u_1} & \cdot & \cdot & \cdot \\ x_5 & \frac{\partial x_5}{\partial u_1} & \cdot & \cdot & \cdot \end{vmatrix} = 0.
$$

In this equation δu and du appear symmetrically and therefore the correspondence between the planes defined by δu and the lines defined by du is symmetric.

When $\delta u = du$, we have

$$
(5') \qquad \left| x_1 \ \frac{\partial x_1}{\partial u_1} \ \frac{\partial x_1}{\partial u_2} \ \frac{\partial x_1}{\partial u_3} \ \Sigma \frac{\partial^2 x_1}{\partial u_i \partial u_j} du_i du_j \right| = 0.
$$

This new equation defines a quadric cone of directions in the tangent π_3. The planes corresponding to the lines of this cone will contain their corresponding line and are tangent planes to the cone. The plane corresponding to any line (direction) through the vertex of the cone is the polar reciprocal of the line with respect to the cone. These properties can be most easily seen by cutting π_3 by an S_3 which does not pass through x. Then in the plane of intersection δu_i may be considered as the coordinates of a line and du_i as the coordinates of a point. Equation (5) defines a polar reciprocation from which the above statements follow if we replace lines of (5′) by points of a conic and planes tangent to (5′) by lines tangent to the conic *ecc*.

The tangents to the directions defined by the cone (5′) will envelope curves on V_3 which have the following properties:

(1) *The osculating planes to such a curve are tangent* to* V_3.

(2) *The tangents to these curves have three point contact with the hypersurface.*

In order to establish these theorems we consider three infinitely near points of the hypersurface which we shall represent in hyperplanar coordinates.† The expression

* A plane is said to be tangent to a hypersurface if it lies in the tangent hyperplane and passes through the point of contact.

† See Segre: Su una classa di Superficie ecc. Atti di Torino, 1907.

$$f = (\xi, x)$$

will be taken to represent the point x. The quantities ξ are hyperplanar coordinates and are not functions of u_1, u_2, u_3. With this notation the three points infinitely near are

$$f = (\xi, x) = 0, \tag{6}$$

$$f_1 du_1 + f_2 du_2 + f_3 du_3 = 0, \tag{7}$$

$$\sum_{ik}^{1,2,3} f_{ik} du_i du_k + \sum_{1}^{3} f_i d^2 u_i = 0 \tag{8}$$

where

$$f_i = \frac{\partial f}{\partial u_i}, \quad f_{ij} = \frac{\partial^2 f}{\partial u_i \partial u_j}.$$

The tangent π_3 is determined by the points

$$f = 0, \quad f_1 = 0, \quad f_2 = 0, \quad f_3 = 0.$$

Points (6) and (7) already lie in this π_3 and we see that (8) is a point on the line joining

$$\Sigma f_{ij} du_i du_j = 0, \quad \Sigma f_i d^2 u_i = 0,$$

the second of which we see always lies in the tangent π_3. Hence the plane determined by (6), (7), (8) will lie in the tangent π_3 if the five points

$$f = 0, \quad f_1 = 0, \quad f_2 = 0, \quad f_3 = 0, \quad \Sigma f_{ik} du_i du_k = 0 \tag{T}$$

lie in this π_3. This condition is exactly equation (5′). Hence

Through each point of V_3 pass ∞^1 curves for each of which the osculating plane is tangent to V_3.

To prove the second property, that is, that tangents to these curves have three point contact with V_3 we will investigate the condition in order that the three points (6), (7), (8) lie in the same line. It will be necessary to show that if these three points lie in a line the five points (T) will lie in a π_3 and conversely if the five points (T) lie in a π_3 (6), (7), (8) are collinear. The first is established as follows. (8) represents a point on the line joining the two points $\Sigma f_{ik} du_i du_k = 0$, $\Sigma f_i d^2 u_i = 0$. This second point is some point of the plane determined by $f_1 = f_2 = f_3 = 0$. The point (7) is also a point of this plane. Then the line joining (7) and (8) must lie in the S_3 determined by the four points $\Sigma f_{ik} du_i du_k = 0$, $f_1 = 0$, $f_2 = 0$, $f_3 = 0$, and if the points (6), (7), (8) are collinear then $f = 0$ must also lie in this same S_3. Equation (5′) is the condition that these five points should lie in the same S_3. Conversely if the above five points lie in an S_3 then $d^2 u_i$ can be determined so that the point (8) will lie on the line

joining (6) and (7). For if d^2u_i are allowed to vary (8) will generate a plane in π_3 and the values of d^2u_i which will give the point where the line joining (6) and (7) cuts this plane will be the values desired. Hence:

The curves tangent to the elements of the cone (5′) are such that their tangents have three point contact with V_3.

If we trace on V_3 any curve c, the planes conjugate to the tangent lines of the curve will envelope a developable* surface.

This is evident from the fact that the tangent π_3's to V_3 in points of c, admit as envelope a system of ∞^1 planes. The envelope of this system of planes is a system of ∞^1 lines which touch a curve.

2. **Asymptotic Directions.**—The ∞^1 directions defined by (5′) can be looked upon as directions along which the tangent hyperplane π_3 has higher order contact with V_3. In fact along these directions π_3 contains points of V_3 infinitely near to the second order, since it contains the osculating planes to the curves in this direction. Let us now examine if there are directions along which π_3 will have contact of third order, that is the osculating S_3 of curves in this direction will coincide with π_3. The osculating S_3 of a curve on V_3 passing through a point O given by (6) is defined by (6), the points (7), (8) and (9)

$$\Sigma f_{ijk} du_i du_j du_k + 3\Sigma f_{ik} du_i d^2u_k + \Sigma f_i d^3u_i = 0.$$

If the first three points lie in π_3, it is necessary that

$$\Sigma f_{ik} du_i du_k = 0 \tag{10}$$

also lies in π_3. Writing this in the form

$$(f_{11}du_1 + f_{12}du_2 + f_{13}du_3)du_i + (f_{12}du_1 + f_{22}du_2 + f_{23}du_3)du_2 \\ + (f_{13}du_1 + f_{23}du_2 + f_{33}du_3)du_3 = 0,$$

and writing the second term of (9) in the form

$$(f_{11}du_1 + f_{12}du_2 + f_{13}du_3)d^2u_1 + (f_{12}du_1 + f_{22}du_2 + f_{23}du_3)d^2u_2 \\ + (f_{13}du_1 + f_{23}du_2 + f_{33}du_3)d^2u_3 = 0, \tag{10′}$$

we see that if (10) lies in π_3 then d^2u_i can be so chosen that this latter point also will lie in it. Since $\Sigma f_i d^3u_i = 0$ lies in π_3, in order for the four points to lie in π_3, it remains only for

$$\Sigma f_{ijk} du_i du_j du_k = 0$$

to lie in it. Then we have the condition (5′) and

* A developable surface here used is a surface which has the same tangent plane at every point of a line.

$$
(11) \qquad \begin{vmatrix}
x_1 & \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} & \Sigma \frac{\partial^3 x_1}{\partial u_1 \partial u_2 \partial u_3} du_1 du_2 du_3 \\
x_2 & \frac{\partial x_2}{\partial u_1} & \cdot & \cdot & \cdot \\
x_3 & \frac{\partial x_3}{\partial u_1} & \cdot & \cdot & \cdot \\
x_4 & \frac{\partial x_4}{\partial u_1} & \cdot & \cdot & \cdot \\
x_5 & \frac{\partial x_5}{\partial u_1} & \cdot & \cdot & \cdot
\end{vmatrix} = 0.
$$

Then the directions along which the osculating S_3 to the curve coincides with π_3 are defined by the values of $du_1 : du_2 : du_3$ which satisfy (5′) and (11). Hence

Through a point O on V_3 pass six directions along which curves can be drawn whose osculating S_3 coincides with the tangent π_3 at O.*

We saw that equation (5′) expressed the condition that the three infinitely near points (x), (dx), (d^2x) should be on the same line, or that the points (6), (7), (8) which determine the plane of these three points should be on the same line. Equation (11) expresses that the point (9) also lies on the same line. For (9) is some point in the plane determined by the three points

$$\Sigma f_{ijk} du_i du_j du_k = 0, \quad \Sigma f_{ij} du_i d^2u_j = 0, \quad \Sigma f_i d^3u_i = 0.$$

The second and third points lie on the line determined by (6), (7), (8) for particular values of d^2u_i and d^3u_i. Then if (9) is to lie on this line, it is only necessary to have $\Sigma f_{ijk} du_i du_j du_k = 0$ on the line which condition is expressed by equation (11). Hence

Through each point of a hypersurface pass six directions such that the tangent line to curves in this direction has four point contact with the hypersurface.

* This can also be seen by expanding the coordinates about the point O. The point $x+dx$ corresponds to the value $u+du$ of the parameters; then

$$x + dx = x + \Sigma \frac{\partial x}{\partial u} du + \frac{1}{2} \Sigma \frac{\partial^2 x}{\partial u_i \partial u_j} du_i du_j + \frac{1}{6} \Sigma \frac{\partial^3 x}{\partial u_i \partial u_j \partial u_k} du_i du_j du_k + \cdots.$$

The tangent hyper-plane contains $x+dx$ to first order infinitesimals and the cone defined by (5′) to second order and if the tangent hyperplane contains

$$\Sigma \frac{\partial^3 x}{\partial u_i \partial u_j \partial u_k} du_i du_j du_k$$

it will contain $x+dx$ to third order infinitesimals.

These curves form the real generalization of asymptotic curves on a surface in ordinary space.

If we expand equation (5′) in terms of du_i we have

(5″) $Pdu_1^2 + Qdu_2^2 + Rdu_3^2 + 2Ldu_1du_2 + 2Mdu_1du_3 + 2Ndu_2du_3 = 0$

as the equation of the cone of principal directions where

$$P = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x_1}{\partial u_1^2} \right|, \quad Q = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x_1}{\partial u_2^2} \right|,$$

$$R = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x_1}{\partial u_3^2} \right|, \quad L = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x_1}{\partial u_1 \partial u_2} \right|,$$

$$M = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x}{\partial u_1 \partial u_3} \right|, \quad N = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x_1}{\partial u_2 \partial u_3} \right|.$$

This cone will degenerate if

$$\begin{vmatrix} P & L & M \\ L & Q & N \\ M & N & R \end{vmatrix} = 0. \tag{12}$$

Since P, Q, R, . . . are functions of the three parameters, usually on V_3 there is a surface along which the cone (5′) will factor.

3. **Special Hypersurfaces.**—Let us examine what happens to this cone of principal directions for two particular kinds of hypersurfaces:

(1) A hypersurface generated by ∞^1 planes. In this case the planes will all touch a fixed curve. For two consecutive planes will intersect in a point and the planes all touch the locus of this point.*

The equation of the hypersurface can then be written

$$x_i = \lambda_i(u_1) + \lambda_i'(u_1)u_2 + \mu_i(u_1)u_3, \tag{H}$$

where the accent indicates differentiation with respect to u_1. From this equation it is at once evident that

$$Q = R = N = 0,$$

but that P, L, M are different from zero. Hence the cone of principal directions degenerates into two distinct planes. It is evident that one of the planes is the generator passing through the point of contact. The equations of the two planes can be written down at once. From (5″)

$$Pdu_1^2 + 2Ldu_1du_2 + 2Mdu_1du_3 = 0.$$

* See Segre: Preliminari di una teoria delle varieta luoghi di spazi. Rendiconti di Palermo, vol. XXX, p. 87.

The two planes then are

$$(G) \qquad du_1 = 0$$

$$(A) \qquad P du_1 + 2L du_2 + 2M du_3 = 0.$$

From the first we have at once the equation of the generator

$$u_1 = \text{const.}$$

The direction of the line of intersection of these two planes is

$$\frac{du_2}{du_3} = -\frac{M}{L}.$$

Substituting the values from (H) in the expression for L and M we have

$$L = |\lambda_1 \mu_1' \lambda_1' \mu_1 \lambda_1''| u_3,$$

$$M = |\lambda_1 \lambda_1'' \lambda_1' \mu_1 \mu_1'| u_2.$$

Hence the above direction reduces to

$$\frac{u_2}{u_3} = \frac{du_2}{du_3}.$$

This defines the direction of the line in a plane $u_1 = \text{const.}$ joining the point (u_2, u_3) to the point $u_2 = 0$, $u_3 = 0$. [In the generator which passes through the point in question u_2, u_3 are the non-homogeneous coordinates of the points. The point where the generator touches the directrix (the envelope of the planes) has coordinates $(0, 0)$.] Hence the line of intersection of the generator (G) with (A) always passes through the point where (G) is tangent to the directrix,

$$(C) \qquad x_i = \lambda_i(u_1).$$

If we write the equation of the tangent S_3 at the point (u_1, u_2, u_3) in the form

$$|\lambda_1 u_2 \lambda_1'' + u_3 \mu_1' \quad \lambda_1' u_1 x_1| = 0,$$

we see that if u_1 is held fixed and $u_2 : u_3$ is constant the tangent S_3 will be the same for the whole line thus determined. Hence a tangent S_3 to V_3 (locus of ∞^1 planes) is tangent the whole length of a line passing through the intersection of (G) and (C).*

Then the planes (A) corresponding to points on a line passing through the intersections of (G) and (C) form a pencil in the tangent S_3.

* The pencil of tangent hyperplanes passing through (G) are projective with the pencil of lines of contact in (G). This correspondence is an extension of that of Chasles for ruled surfaces in S_3. This is however only a special case of the correspondence mentioned by Segre, Preliminari di una teoria ecc. Rendiconti di Palermo, vol. XXX.

(2) If the system consists of the osculating planes of a curve

$$(C) \qquad x_i = \lambda_i(u_1),$$

then the equation of the V_3 becomes

$$x_i = \lambda_i(u_1) + u_2\lambda_i'(u_1) + u_3\lambda_i''(u_1).$$

In this case we see at once

$$Q = R = L = M = N = 0.$$

Hence the two planes of principal directions coincide. This hypersurface is a hyperdevelopable that is the envelope of ∞^1 hyperplanes. Consecutive planes of the system intersect in a line. The surface formed by the lines of intersection which we will call the singular developable might be compared to the cuspidal edge of a developable in S_3. An S_3 will cut V_3 in a developable and the edge of regression is the curve in which S_3 cuts this singular developable.

4. **Complement of Conjugate Directions.**—To the correspondence between the plane of intersection of two consecutive tangent spaces and the line joining their points of contact one obtains a complement by considering the line of intersection of three successive tangent spaces and the plane determined by the three points of tangency.*

In this way lines in the tangent space π_3 at a point O are made to correspond to the planes which pass through O. This correspondence is (1, 1). Some of the properties can be obtained immediately, e. g., to the planes passing through a line t which is tangent to V_3 in O will correspond the lines of the plane p the conjugate of t in the correspondence of section 1.

In order to obtain the equations of the correspondence, we shall consider the tangent cone drawn to V_3 from a line of π_3. Here a cone means ∞^1 planes which have a line in common. Then the osculating planes of the curve of contact of V_3 with the cone will correspond to the line which forms the vertex of the cone. We will take the point (1, 0, 0, 0, 0) as O and take the hyperplane $x_5 = 0$ as π_3. Let the equation of V_3 in homogeneous coordinates be

$$f(x_1, x_2, x_3, x_4, x_5) = f = 0.$$

The π_3 tangent at $(x_1, x_2, x_3, x_4, x_5)$ is

$$\Sigma f_i x_i = 0$$

(f_i is used for the derivative $\partial f/\partial x_i$). And if $x_5 = 0$ is to be tangent at the point (1, 0, 0, 0, 0) then

* See Segre: Complementi alla teoria delle tangenti coniugati di una superficie. Rendicont dei Lincei, vol. XVII, p. 405.

$$f_1 = f_2 = f_3 = f_4 = 0 \tag{13}$$

at this point. Let $(y_1, y_2, y_3, y_4, y_5)$ and $(z_1, z_2, z_3, z_4, z_5)$ denote two points in π_3. The curve of contact of the tangent cone which has the line joining these two points for vertex will be the intersection of the three hypers

$$\begin{aligned} f &= 0, \\ \Sigma y_i f_i &= 0, \\ \Sigma z_i f_i &= 0. \end{aligned} \tag{K}$$

The line of our correspondence is the line joining the points y and z and the corresponding plane is the osculating plane of (K) at O. We shall use the Plücker coordinates for the lines of π_3 $(x_5 = 0)$ which are the determinants of the matrix

$$\begin{Vmatrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{Vmatrix} \tag{L}$$

and which we shall indicate as usual by

$$p_{ik} = y_i z_k - y_k z_i.$$

We shall use as the coordinates of the planes the determinants of the matrix of the three points O, dx/dt, d^2x/dt^2 where dx/dt and d^2x/dt^2 are determined from the equations of (K). (Here it is assumed that t is the parameter in terms of which we have expressed the coordinates of (K)). Now as the coordinates of O are $(1, 0, 0, 0, 0)$ the coordinates of the planes passing through O are the three row determinants of the matrix

$$\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{dx_1}{dt} & \frac{dx_2}{dt} & \frac{dx_3}{dt} & \frac{dx_4}{dt} & \frac{dx_5}{dt} \\ \frac{d^2x_1}{dt^2} & \frac{d^2x_2}{dt^2} & \frac{d^2x_3}{dt^2} & \frac{d^2x_4}{dt^2} & \frac{d^2x_5}{dt^2} \end{Vmatrix}.$$

Thus the coordinates of the planes are

$$q_{ik} = \frac{dx_i}{dt}\frac{d^2x_k}{dt^2} - \frac{dx_k}{dt}\frac{d^2x_i}{dt^2} \qquad (i, k = 2, 3, 4, 5). \tag{P}$$

To obtain the equations of the correspondence it is only necessary to find the ratio of the values of dx/dt and d^2x/dt^2. In order to do this we shall make use of the relation

$$\Sigma x_i = 1,$$

which renders the coordinates homogeneous.

Then to determine dx we have

$$\sum_i \frac{dx}{dt_i} = 0, \qquad \sum_i f_i \frac{dx}{dt_i} = 0,$$

$$\sum_{ik} y_i f_{ik} \frac{dx}{dt_k} = 0, \quad \sum_{jk} z_i f_{ik} \frac{dx}{dt_k} = 0,$$

from which we have

$$\frac{dx_1}{dt} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ f_2 & f_3 & f_4 & f_5 \\ \Sigma y_i f_{2i} & \cdot\ \cdot & \cdot\ \cdot & \Sigma y_i f_{5i} \\ \Sigma z_i f_{2i} & \cdot\ \cdot & \cdot\ \cdot & \Sigma z_i f_{5i} \end{vmatrix},$$

and expanding according to the minors of the last two rows,

$$\frac{dx_1}{dt} = \pm \sum_{i,j=1}^{i,j=4} \sum_{k,l=2}^{k,l=5} \begin{vmatrix} f_{ik} & f_{jk} \\ f_{il} & f_{jl} \end{vmatrix} (f_m - f_n) p_{ij},$$

where m, n are the remaining numbers of 2, 3, 4, 5 after k, l are chosen. The other derivatives can now be written down from symmetry

$$\frac{dx_h}{dt} = \pm \sum_{i,j=1}^{i,j=4} h \sum_{k,l=1}^{k,l=5} \begin{vmatrix} f_{ik} & f_{jk} \\ f_{il} & f_{jl} \end{vmatrix} (f_m - f_n) p_{ij},$$

where the letter h written in front of the second summation sign indicates that from the sequence 1, 2, 3, 4, 5 the number h is to be omitted. As above m, n are the remaining numbers of 1, 2, 3, 4, 5 after h, k, l are chosen. The expressions for d^2x/dt^2 can be obtained at once by differentiating. They are

$$\frac{d^2x_h}{dt^2} = \sum_{s=1}^{s=5} \sum_{i,j=1}^{i,j=5} h \sum_{k,l=1}^{k,l=5} \begin{vmatrix} f_{iks} & f_{jk} \\ f_{ils} & f_{jl} \end{vmatrix} (f_m - f_n) p_{ik} \frac{dx_s}{dt}$$

$$+ \sum_{s=1}^{s=5} \sum_{i,j=1}^{i,j=4} h \sum_{k,l=1}^{k,l=5} \begin{vmatrix} f_{ik} & f_{jks} \\ f_{il} & f_{jls} \end{vmatrix} (f_m - f_n) p_{ik} \frac{dx_s}{dt}$$

$$+ \sum_{s=1}^{s=5} \sum_{i,j=1}^{i,j=4} h \sum_{k,l=1}^{k,l=5} \begin{vmatrix} f_{ik} & f_{jk} \\ f_{il} & f_{jl} \end{vmatrix} (f_{ms} - f_{ns}) p_{ik} \frac{dx_s}{dt}.$$

Now making use of relations (13) we see that

$$\frac{dx_5}{dt} = 0,$$

since

$$\Sigma f_i \frac{dx_i}{dt} = 0.$$

Then the expressions for the derivatives become

$$\frac{dx_k}{dt} = \pm \sum_{i,j=1}^{i,j=4} h \sum_{k,l=1}^{k,l=5} \begin{vmatrix} f_{ik} & f_{jk} \\ f_{il} & f_{jl} \end{vmatrix} f_5 p_{ij},$$

and

$$\frac{d^2x_k}{dt^2} = \pm \sum_{s=1}^{s=5} \sum_{i,j=1}^{i,j=4} h \sum_{k,l=1}^{k,l=5} \left[\begin{vmatrix} f_{iks} & f_{jk} \\ f_{ils} & f_{jl} \end{vmatrix} f_5 + \begin{vmatrix} f_{ik} & f_{jks} \\ f_{il} & f_{jls} \end{vmatrix} f_5 + \begin{vmatrix} f_{ik} & f_{jk} \\ f_{il} & f_{jl} \end{vmatrix} (f_{ms} - f_{ns}) \right] p_{ij} \frac{dx_s}{dt}.$$

From these values the coordinates q_{ik} of the osculating planes can be calculated and we see at once that the correspondence is cubic.

5. Proceeding with the correspondence between the lines p_{ij} of π_3 and the planes q_{ik} passing through O, let us take O and three successive points of a curve passing through O and draw the four tangent spaces to V_3 at these points and put in the condition that the four points lie in the same plane q_{ik} and that the four tangent hyperplanes intersect in the same line p_{ij} of π_3. In this case the curve of contact of the cone circumscribed to V_3 with the line p_{ij} for vertex will have the plane q_{ik} for hyperosculating plane. Then the four points

$$x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \frac{d^3x}{dt^3},$$

will be connected by a linear relation

$$Ax + B\frac{dx}{dt} + C\frac{d^2x}{dt^2} + D\frac{d^3x}{dt^3} = 0,$$

which will require the vanishing of two four row determinants of the matrix

$$\begin{Vmatrix} x & \dfrac{dx}{dt} & \dfrac{d^2x}{dt^2} & \dfrac{d^3x}{dt^3} \end{Vmatrix}$$

From the expression for d^2x/dt^2 it is at once seen that d^3x/dt^3 is of order three in the line coordinates p_{ij} and hence the determinants of the above matrix are of order six in p_{ij}. Therefore: *There is a congruence of lines of order 36 in π_3 such that any tangent cone having one of these lines for vertex will have a hyperosculating plane at the point of tangency of the π_3.*

By the same method of reasoning it is easily seen that in π_3 there are $6^2 \cdot 7^2$ lines such that the curves of contact of the tangent cones drawn to V_3 from these lines has in O a plane which has contact of order four, that is O and four successive points lie in one plane.

6. **Hypersurfaces in S_n.**—The preceding discussions can be applied to V_{n-1} in S_n without change. The lines passing through a point O of V_{n-1} which have three point contact with V_{n-1} form a quadric cone in the tangent S_{n-1}. This cone is the cone of united elements of the correspondence formed by taking the line joining two successive points of V_3 and making it correspond to the S_{n-2} of intersection of the tangent spaces to V_3 at those points. An S_{n-2} passing through O (and tangent to V_3) and its corresponding line are polar reciprocals with respect to the above cone. The osculating plane to the curves in the direction of the elements of the cone will lie in the tangent S_{n-1} at O. The directions along which the osculating S_3 of the curve lies in the tangent S_{n-1} form a cone of order 3 and dimensions $n-3$. The elements have four-point contact with V_{n-1}. The directions along which the osculating S_4 to the curve lies in in the tangent S_{n-1} form a cone of order 4 and dimensions $n-4$ and the elements have five-point contact with V_{n-1}, etc. Finally there are $n-1$ directions such that the osculating S_{n-1} to the curves will coincide with the tangent S_{n-1}. These directions have $(n-1)$-point contact with V_{n-1} and form a generalization of asymptotic directions.

If we consider the curves on V_{n-1} which pass through O and take the tangent spaces S_{n-1} at three successive points and the plane which contains the three points then we establish a (1, 1) correspondence between the planes passing through O and the spaces S_{n-3} contained in the tangent S_{n-1} to V_3 at O. The correspondence is cubic as in S_4. There are ∞^{n-2} S_{n-3}'s such that the curve of intersection of the tangent cone drawn to V_3 from them has in O a hyperosculating plane. These S_{n-3}'s form an $(n-2)$ parameter family of order 6^{n-2}. In the tangent S_{n-1} there are $6^{n-2}\cdot 7^{n-2}$ S_{n-3}'s such that the tangent cone to V_3 from one of them will be tangent along a curve which has in O a plane which has fourth order contact.

Massachusetts Institute of Technology,
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A THIRD GENERALIZATION OF THE GROUPS OF THE REGULAR POLYHEDRONS.

By G. A. Miller.

§ 1. Introduction.

On November 10, 1856, Sir William R. Hamilton presented before the Irish Academy a paper entitled "A new system of roots of unity" * in which he pointed out the interesting fact that if s_1, s_2 represent two operators, which obey the associative but not the commutative law of multiplication, the three sets of three equations,

$$s_1^2 = s_2^3 = (s_1 s_2)^r = 1; \qquad r = 3,\ 4,\ 5,$$

define the groups of movements of the regular polyhedrons. The case when $r = 5$ seems to have interested him especially and he denoted the resulting group in this special case by "Icosian Calculus," observing that all these results may be represented geometrically on the regular icosahedron or on the regular dodecahedron. About a quarter of a century later Dyck rediscovered the same results and put the whole matter in a somewhat clearer form from the standpoint of abstract groups.† These relations are so simple and admit such a variety of geometric interpretations that they have become classic.

Two generalizations of these Hamiltonian relations were developed about a quarter of a century after the publications by Dyck, or about half a century after Hamilton had started investigations in this direction. In the earlier of these generalizations ‡ the groups generated by s_1, s_2 when two of the three Hamiltonian relations are replaced by a single one, without changing the third, were investigated, and the possible groups were determined. It was assumed throughout this article that the generating operators were not commutative. The special cases when these operators are commutative were considered incidentally in a later article.§ In the latter article a few errors relating to the first generalizations of the icosahedral group were also corrected.

A second generalization of the given Hamiltonian relations appeared

* Hamilton, Proceedings of the Royal Irish Academy, vol. 6 (1853–7), p. 415.

† Dyck, Mathematische Annalen, vol. 22 (1883), p. 82.

‡ Transactions of the American Mathematical Society, vol. 8 (1907), p. 1.

§ Quarterly Journal of Mathematics, vol. 41 (1910), p. 171.

in two articles. The former of these was devoted to a second generalization of the relations defining the tetrahedral and the octahedral groups* while the latter confined itself to a second generalization of the icosahedral group.† In these second generalizations the third Hamiltonian relation $(s_1s_2)^r = 1$ was replaced by $(s_1s_2)^r = (s_2s_1)^r$, while the other two relat ons were replaced by a single one, just as in the first generalizations. While only a small number of different groups involve two generators which satisfy the former of these generalized conditions, there is always an infinite number of such groups when the latter generalized relations are assumed to be the only conditions impo ed on the two generators.

The generalizations considered in the present paper are more direct than those previously considered as they are obtained from the Hamiltonian relations by omitting the condition that the given powers of s_1, s_2, s_1s_2 are equal to unity. That is, we consider relations of the general type

$$s_1^2 = s_2^3 = (s_1s_2)^r, \qquad r = 3,\ 4,\ 5.$$

Such relations are evidently equivalent to the two conditions $s_1^2 = s_2^3$, $s_1^2 = (s_1s_2)^r$ and it will be proved that they are satisfied by the two generators of at least two and at most four groups for each value of r. The simplicity of these results seems to justify the hope that they may find extensive applications.

It should perhaps be emphasized that the results here obtained are considerably simpler than those obtained from the other two generalizations that have been noted, and hence the present paper has closer contact with the original developments by Hamilton than these earlier generalizations have. The reason that the present generalizations were not developed first is that the writer did not foresee that they would lead to simple results, and did not notice an easy approach to this problem when the other generalizations were taken up.

The method employed in § 2 is quite different from that used in the following sections. This change is due to the fact that it was thought that the former method would give a deeper insight into the problem and the considerations of § 2 were sufficiently simple to employ a more general method of work than seemed feasible in the other sections where the considerations become more complex. It may however be well to indicate here how to prove the main results of § 2 by the more special methods employed later.

Starting with the equations $s_1^2 = s_2^3 = (s_1s_2)^3$ we observe at once that s_1, $s_2^{-1}s_1s_2$ have a common square, since s_1^2 is invariant under the group G generated by s_1, s_2. Hence $s_1s_2^{-1}s_1^{-1}s_2$ is transformed into its inverse by s_1

* American Journal of Mathematics, vol. 32 (1910), p. 65.

† Quarterly Journal of Mathematics, vol. 41 (1910), p. 168.

according to the theorem: If two operators have a common square the product of one and the inverse of the other is transformed into its inverse by each of these operators.* By means of the equations,

$$s_1 = s_2 s_1 s_2 s_1 s_2, \qquad s_2^2 = s_1 s_2 s_1 s_2 s_1,$$

which are equivalent to the given conditional equations, it is easy to verify the following relations:

$$(s_1 s_2^{-1} s_1^{-1} s_2)^2 = (s_2 s_1 s_2^2)^2 = s_2 s_1 s_2^3 s_1 s_2^2 = s_1^6.$$

As s_1^6 is both invariant under s_1 and also transformed into its inverse by s_1, its order divides 12. Hence the order of s_2 must divide 18 and the order of G divides 72.†

After having proved that the order of G divides 72 whenever G is generated by two operators which satisfy the given conditions, it is comparatively easy to determine the total number of groups which may be generated by two such operators, and these details are given in the following section. In a similar way it may be observed that when

$$s_1^3 = s_2^3 = (s_1 s_2)^2; \text{ or } s_1^2 = s_2 s_1 s_2, \; s_2^2 = s_1 s_2 s_1$$

the two operators $s_1 s_2$, $s_2 s_1$ have a common square, and hence

$$(s_1 s_2 s_1^{-1} s_2^{-1})^2 = s_1^{-6}(s_1 s_2 s_1^2 s_2^{-1})^2 = s_1^{-6}(s_1 s_2^2 s_1)^2 = s_1^{-6}(s_1^2 s_2 s_1^2)^2$$
$$= s_1^{-3} s_1^2 s_2 s_1 s_2 s_1^2 = s_1^3.$$

Since s_1^3 is invariant under $s_1 s_2$ and also transformed into its inverse by $s_1 s_2$, the order of s_1 divides 6 and the order of G divides 24. Hence it is again very easy to complete the determination of all the possible groups which can be generated by two operators satisfying the given conditions. These special developments appear in the following section.

It may be added that if a set of conditions in the form of equations is given this set is always satisfied by the assumption that each of the operators is the identity. This trivial special case is not generally mentioned in what follows as it is so evident and exists always. Hence it must generally be assumed in the following theorems that at least one of the operators under consideration is not the identity. This is so often done in group theory literature that it scarcely calls for justification in an article. The types of alternative proofs suggested for the results of § 2 may also be employed to prove many of the results of the following sections.

* Archiv der Mathematik und Physik, vol. 9 (1905), p. 6.

† This upper limit of the order of G is a direct result of the well known theorem that two non-commutative operators s_1, s_2 which satisfy the equations $s_1^2 = s_2^3 = (s_1 s_2)^3 = 1$ must generate the tetrahedral group.

§2. Generalization of the tetrahedral group.

If the two non-commutative generators s_1, s_2 of a group G satisfy the conditions

$$s_1^2 = s_2^3 = (s_1s_2)^3$$

it results directly that the cyclic group generated by s_1^2 is the central of G, and that the quotient group of G with respect to its central (the central quotient group of G) is the tetrahedral group. To the invariant subgroup of order 4 in this central quotient group there corresponds a subgroup of G whose operators of odd order are in its central and hence this subgroup is the direct product of a cyclic group of odd order and a group of order 2^α. We proceed to prove that the latter is either the four group or the quaternion group. In fact, this follows directly from the necessary property of this group that it involves three cyclic subgroups of order $2^{\alpha-1}$ which are conjugate under G. It is well known that there are only two groups of order 2^α which contain three cyclic subgroups of order $2^{\alpha-1}$ and that the groups of order 2^α which contain more than one cyclic subgroup of order $2^{\alpha-1}$ are conformal with abelian groups whenever $\alpha > 3$.

From the preceding paragraph it follows directly that the order of G cannot be divisible 16 whenever G is generated by s_1, s_2 subject to the given conditions. It is also easy to see that G cannot involve a subgroup of half its own order, since such a subgroup would involve exactly half the operators of G which correspond to each operator in its central quotient group. In particular, this subgroup would involve half of the central of G, but this half central could not involve the square of any operator corresponding to an operator of order 2 in the central quotient group and hence G cannot involve a subgroup of half its own order.

As there is only one group of order 24 which does not contain a subgroup of order 12, it results from the preceding paragraph that G must be this non-twelve group of order 24 when the order of s_1 is 4. From the properties of this group of order 24 it is clear that it can be generated by two operators of orders 4 and 6 respectively, which satisfy the conditions imposed on s_1 and s_2. That is, when s_1 is of order 2, G is the tetrahedral group, and when s_1 is of order 4, G is the non-twelve group of order 24. We shall soon be able to prove that the order of G must divide 72 and that the two non-commutative operators s_1, s_2 must generate one of four groups when they satisfy the given conditions.

To establish this fact, it is convenient to make use of a theorem which has very wide applications and may be stated as follows: *If $s_1, s_2, \cdots, s_\rho$ is a complete set of conjugates, in order, under an operator t and if the continued product of this set of conjugates, in order, is the identity then will $(s_\alpha t)^\rho = t^\rho$,*

where α is any one of the numbers 1, 2, $\cdots$, ρ. Since $t^{-1}s_\alpha t = s_{\alpha+1}(\alpha = 1, 2, \cdots, \rho - 1)$ it results that $s_\alpha t = ts_{\alpha+1}$. Hence $(s_\alpha t)^\rho = s_\alpha s_{\alpha-1} \cdots s_{\alpha+1}t^\rho$, where $\alpha - 1, \alpha - 2, \cdots$ are to be replaced by their least positive residues modulo ρ except that 0 is replaced by ρ. From the fact that $s_1s_2 \cdots s_\rho = 1$ it results that $s_\alpha s_{\alpha-1} \cdots s_{\alpha+1} = 1$ and hence the theorem is proved. It is clear that the given theorem remains true when the set $s_1, s_2, \cdots, s_\rho$ represents more than one complete set of conjugates, in order, under t, since the given proof does not depend upon the fact that the operators $s_1, s_2, \cdots, s_\rho$ are distinct. Other generalizations of the given theorem at once suggest themselves but for our present purpose it is convenient to leave it in the special form in which it has been stated.

It has been observed above that s_1 may be regarded as the direct product of s_2^{3a} and an operater s_1' which is either of order 2 or of order 4, and that s_2 transforms s_1' into three conjugates s_1', s_2', s_3' whose continued product is the identity when s_1' is of order 2, since these three operators of order 2 and the identity constitute the four group. This continued product must also be the identity when s_1' is of order 4 since $(s_1's_2)^3$ is of even order in this case. Hence we have that $s_1' s_2' s_3' = 1$ in all cases, and if we combine this with the theorem of the preceding paragraph it results that the conditions given at the beginning of this section may be replaced by

$$s_1'^2 s_2^{6a} = s_2^3 = (s_1's_2)^3 s_2^{9a} = s_2^{9a+3}.$$

Hence $s_2^{9a} = 1$, and as α and the order of s_2 have only 2 or 1 as their highest common factor it results that $s_2 = 1^{18}$.

To prove that the order of s_2 may be 18 it may be convenient to begin with the case when s_1, s_2 are commutative and hence the given conditions reduce to

$$s_1^2 = s_2^3 = s_1^3 s_2^3.$$

From these conditions it results directly that $s_1^3 = 1$, and hence $s_2^9 = 1$. Moreover, if s_2 is an operator of order 9 and $s_1 = s_2^6$, it is evident that s_1, s_2 satisfy the given conditions and generate the cyclic group of order 9. As it has been observed that s_1, s_2 may be so chosen that their orders are 4 and 6 respectively and that they generate the non-twelve group of order 24, it results directly that we may associate with these two non-commutative operators two commutative ones t_1, t_2 (which are also commutative with s_1, s_2) of orders 3 and 9 respectively so that s_1t_1, s_2t_2 are two operators of order 12 and 18 respectively which satisfy the given conditions. In the same way we see that the orders of s_1, s_2 may be 6 and 9 respectively and hence there results the theorem: *If two non-commutative operators s_1, s_2 satisfy the two conditions $s_1^2 = s_2^3 = (s_1s_2)^3$ they must generate one of the following four groups: the tetrahedral group, the non-twelve group of order* 24,

or the groups obtained by establishing a tris-isomorphism between each of these groups and the cyclic group of order 9. When s_1, s_2 are commutative and satisfy these conditions they generate either the cyclic group of order 9 or the group of order 3.*

Closely related to the generalization considered above is the following:

$$s_1^3 = s_2^3 = (s_1s_2)^2.$$

That this does not lead to the same category as the set considered above is evident from the fact that if s_1, s_2 are commutative and satisfy these conditions they generate the group of order 3, while in the preceding case they could also generate the group of order 9. We shall again begin with the case when these generating operators are non-commutative, and hence G is a group whose central quotient group is the tetrahedral group. Just as in the preceding case we observe that, when s_1s_2 is of order 2 or 4, G is the tetrahedral group or the non-twelve group of order 24, and that G cannot involve a subgroup of index 2. We proceed to prove that G must be one of these two groups when s_1, s_2 are non-commutative and satisfy these conditions.

The subgroup of G which corresponds to the four-group in the central quotient group is the direct product of a group of odd order and either the four-group or the quaternion group, for the same reasons as were given under the preceding conditions. Hence we may assume that $s_1s_2 = s's_1^{3a}$, where s' is either of order 2 or 4 and is commutative with s_1^{3a}. Moreover, a has at most the factor 2 in common with the order of s_1 since s_1^{3a} must generate the group of odd order in the central of G. Hence the following equations:

$$(s_1s_2)^2 = s'^2s_1^{6a} = s_1^3; \quad s_1^{12a} = s_1^6.$$

On the other hand, we have the equations

$$s_1^{-1}\cdot s_1s_2 = s_1^{-1}s'\cdot s_1^{3a}; \quad s_2^6 = s_1^6 = s_1^{-6}s_1^{18a},$$

in accord with the general theorem given above. From these equations it results directly that

$$s_1^{24a} = s_1^{18a}, \text{ or } s_1^{6a} = 1.$$

As the order of s_1 cannot be divisible by 4 it results that the order of s_1 divided 6, and hence the order of s_2 is also a divisor of this number. This

* A tris-isomorphism is one in which the invariant subgroups of index 3 are made to correspond, hence the order of the resulting group is one-third of the product of the orders of the isomorphic groups. When the invariant subgroups of index 2 are made to correspond the isomorphism may be called a dim-isomorphism. Cf. Cayley, Quarterly Journal of Mathematics, vol. 25 (1890), p. 85.

proves that the order of s_1s_2 is either 2 and 4 and completes the proof of the theorem: *If two non-commutative operators satisfy the conditions* $s_1^3 = s_2^3 = (s_1s_2)^2$ *they generate either the tetrahedral group or the non-twelve group of order* 24. *If two commutative operators satsify these conditions they generate the group of order* 3 *unless each of these operators is the identity.*

In speaking of the finite group generated by operators subject to certain cond tions it is customary to think of the largest possible group which these operators can generate under these conditions. If this were done in the present section the two theorems on the generalization of the tetrahedral group could be expressed as follows: If two operators satisfy the conditions $s_1^3 = s_2^3 = (s_1s_2)^2$ they generate the non-twelve group of order 24, and if they satisfy the conditions $s_1^2 = s_2^3 = (s_1s_2)^3$ they generate the group of order 72 formed by establishing a tris-isomorphism between this group of order 24 and the cyclic group of order 9. The theorems as stated above are, however, more definite as they include the cases when s_1, s_2 do not generate the largest possible group subject to the given conditions.

§ 3. Generalizations of the octahedral group.

If the two non-commutative operators s_1, s_2 satisfy the two conditions

$$s_1^3 = s_2^4 = (s_1s_2)^2,$$

the two operators s_2, $s_1^{-1}s_2^2s_1$ have a common square, and hence the product of one of these into the inverse of the other is transformed into its inverse by each of these operators. We may therefore endeavor to find an upper limit of the order of the group G generated by s_1, s_2 by finding a power of this product such that this power is invariant under G. From the fact that this power is both transformed into its inverse and is also invariant under the same operator, it results directly that its order cannot exceed 2. The actual work may be performed as follows.

From the given relations it results directly that

$$s_1s_2s_1 = s_2^3 \quad \text{and} \quad s_2s_1s_2 = s_1^2.$$

Hence it is not difficult to derive the following relations:

$$\begin{aligned}(s_2^2s_1^{-1}s_2^{-2}s_1)^2 &= s_2^2s_1^{-1}\cdot s_2^{-2}s_1s_2^2\cdot s_1^{-1}s_2^{-2}s_1 = s_2^2s_1^{-1}s_2^{-3}s_1^2s_2s_1^{-1}s_2^{-2}s_1\\ &= s_2^2s_1^{-2}s_2^{-1}s_1s_2s_1^{-1}s_2^{-2}s_1 = s_2^2s_1^{-2}s_2^{-2}s_1s_2^{-2}s_1\\ &= s_2^2s_1^{-2}s_2^{-3}s_1^2s_2^{-3}s_1 = s_1^{-9}s_2^2s_1s_2s_1^2s_2s_1\\ &= s_1^{-9}s_2^5s_1s_2s_1 = s_1^{-3}.\end{aligned}$$

This proves that the order of s_1 is a divisor of 6 whenever the given con-

ditions are satisfied, and hence the order of G cannot exceed 48. Moreover, it is easy to see that the non-twelve group of order 24 can be extended by means of an operator of order 4 so as to obtain a group of order 48 which may be generated by two operators satisfying the given condition. Hence the theorem: *If two non-commutative operators fulfil the condition* $s_1^3 = s_2^4 = (s_1s_2)^2$ *they generate either the octahedral group or a group of order* 48 *known as* G_{52}.* *If two commutative operators satisfy these conditions they evidently generate the group of order* 2.

If two non-commutative operators satisfy the two conditions

$$s_1^2 = s_2^4 = (s_1s_2)^3; \quad \text{or} \quad s_1 = s_2s_1s_2s_1s_2,\ s_2^3 = s_1s_2s_1s_2s_1,$$

the two operators, s_2^2, $s_1^{-1}s_2^2s_1$ have again a common square and hence we shall consider the reduced form of $(s_2^2s_1^{-1}s_2^{-2}s_1)^2$ as follows

$$\begin{aligned}(s_2^2s_1^{-1}s_2^{-2}s_1)^2 &= s_2^2s_1^{-1}s_2^{-2}s_1s_2^2s_1^{-1}s_2^{-2}s_1 = s_2^2s_1^{-1}s_2^{-1}s_1s_2s_1s_2^3s_1^{-1}s_2^{-2}s_1\\ &= s_2^2s_1^{-1}s_2^{-1}s_1s_2s_1^2s_2s_1s_2^{-1}s_1 = s_2^2s_1s_2^{-1}s_1s_2^2s_1s_2^{-1}s_1\\ &= s_2^2s_1^2s_2s_1s_2^3s_1s_2^{-1}s_1 = s_1^2(s_2^3s_1)^3s_2^{-4} = s_1^{10}.\end{aligned}$$

Hence it results that the order of s_1 is a divisor of 20 and that the order of G divides 240. If s_1 is actually of order 20 the order of s_2 must be 40, and s_1^{15}, s_1^5 satisfy the given relations. These operators must therefore generate G_{52}, and G must be the direct product of this G_{52} and the group of order 5. On the other hand, if the order of s_1 is 10 the given conditions are again satisfied and G must be the direct product of the octahedral group and the group of order 5. If s_1, s_2 are commutative and satisfy the given conditions they clearly generate either the group of order 5 or the cyclic group of order 10. Hence the theorem: *If two non-commutative operators satisfy the conditions* $s_1^2 = s_2^4 = (s_1s_2)^3$ *they generate one of the following four groups; the octahedral group,* G_{52}, *or the direct product of one of these groups and the group of order* 5. *When two commutative operators satisfy these conditions, they generate the group of order* 2 *or of order* 5, *or the cyclic group of order* 10.

The third and last generalization of the octahedral group to be considered in this connection is given by the equations

$$s_1^2 = s_2^3 = (s_1s_2)^4.$$

From these equations we easily deduce an equivalent system as follows:

$$s_1 = s_2s_1s_2s_1s_2s_1s_2,\ s_2^2 = s_1s_2s_1s_2s_1s_2s_1.$$

We shall proceed again in the same manner as in the two preceding cases observing that $s_1s_2s_1s_2$ and $s_2s_1s_2s_1$ have a common square since $(s_1s_2)^4 = (s_2s_1)^4$ is invariant under G. Hence the following equations

* Quarterly Journal of Mathematics, vol. 30 (1898–9), p. 258.

$$(s_1s_2s_1s_2s_1^{-1}s_2^{-1}s_1^{-1}s_2^{-1})^2 = s_1^{-6}(s_1s_2s_1s_2s_1s_2^2\cdot s_1s_2^2)^2 = s_1^{-16}(s_2^{-1}s_1^{-1}s_2^4s_1s_2^2)^2$$
$$= s_1^{-20}(s_2^2s_1s_2s_1s_2^4s_1s_2s_1s_2^2) = s_1^{-16}s_2^{-1}(s_1s_2)^4s_2 = s_1^{-14}.$$

Hence the order of s_1 divides 28 and the order of G divides 336.

When s_1, s_2 are commutative it is very easy to see that they generate one of the groups of orders 2 and 7, or the cyclic group of order 14. It is therefore evident that G may be the direct product of G_{52} and the group of order 7. Moreover, when s_1 is of order 28 the two operators s_1^{21}, s_2^7 clearly satisfy the conditions imposed on s_1, s_2 and hence $(s_1^{21}, s_2^7) \equiv G_{52}$. The same operators clearly generate the octahedral group when the order of s_1 is 14. From these results we readily derive the following theorem: *If two non-commutative operators satisfy the conditions $s_1^2 = s_2^3 = (s_1s_2)^4$ they generate one of the following four groups: the octahedral group, the group of order 48 known as G_{52}, or the direct product of these groups and the group of order 7. If two commutative operators satisfy these conditions they generate one of the following three groups: the groups of orders 2 and 7, or the cyclic group of order* 14.

It is known that the icosahedral group is generated by s_1, s_2 whenever these two operators satisfy one of the following three sets of three conditions:

$$s_1^3 = s_2^5 = (s_1s_2)^2 = 1, \quad s_1^2 = s_2^5 = (s_1s_2)^3 = 1, \quad s_1^2 = s_2^3 = (s_1s_2)^5 = 1.$$

In the present section we shall consider the possible groups generated by s_1, s_2 when they satisfy one of the following three sets of two conditions:

$$s_1^3 = s_2^5 = (s_1s_2)^2, \quad s_1^2 = s_2^5 = (s_1s_2)^3, \quad s_1^2 = s_2^3 = (s_1s_2)^5.$$

While the number of conditions imposed on s_1, s_2 is thus reduced in each case it will appear that the considerations are not made much more complex thereby. The methods used in the present section are similar to those used in the preceding section and hence they require no further explanations. We shall consider the three cases in the given order.

When $s_1^3 = s_2^5 = (s_1s_2)^2$ it results immediately that

$$s_1^2 = s_2s_1s_2 \quad \text{and} \quad s_2^4 = s_1s_2s_1.$$

Since s_1s_2 and s_2s_1 have a common square we consider the powers of $s_1s_2s_1^{-1}s_2^{-1} = s_1^{-3}s_1s_2s_1^2s_2^{-1} = s_1^{-3}s_1s_2^2s_1$ as follows:

$$(s_1s_2s_1^{-1}s_2^{-1})^2 = s_1^{-6}s_1s_2^2s_1^2s_2^2s_1 = s_1^{-6}s_1s_2s_2^3s_1s_2^3s_1$$

$$(s_1s_2s_1^{-1}s_2^{-1})^3 = s_1^{-9}s_1s_2^3s_1s_2^3s_1^2s_2^2s_1 = s_1^{-9}s_1s_2^3s_1s_2^4s_1s_2^3s_1$$

$$= s_1^{-9}s_1s_2^3s_1^2s_2s_1^2s_2^3s_1 = s_1^{-9}s_1s_2^4s_1s_2^3s_1s_2^4s_1$$

$$= s_1^{-9}s_1^2s_2s_1^2s_2^3s_1^2s_2s_1^2 = s_1^{-9}s_1^2s_2^2s_1s_2^5s_1s_2^2s_1^2$$

$$= s_1^{-6}s_1^2s_2^2s_1^2s_2^2s_1^2$$

$$(s_1s_2s_1^{-1}s_2^{-1})^5 = s_1^{-12}s_1^2s_2^2s_1^2s_2^2s_1^2s_1s_2^3s_1s_2^3s_1 = s_1^3.$$

As s_1^3 is both transformed into its inverse by s_1s_2 and is also invariant under G it results that the order of G divides 120. It is known that there is a group of order 120, known as G_{120}, which is generated by two operators of orders 6 and 10 respectively which satisfy the given conditions,* and hence we have the theorem: *If two operators satisfy the two conditions* $s_1^3 = s_2^5 = (s_1s_2)^2$ *they generate either the icosahedral group or a group of order* 120 *known as* G_{120}. This group of order 120 is the smallest compound perfect group.†

When $s_1^2 = s_2^5 = (s_1s_2)^3$, the two operators s_1, $s_2^{-1}s_1s_2$ have a common square and hence we shall consider the various powers of $s_1s_2^{-1}s_1^{-1}s_2$. These powers may be reduced by means of the equations

$$s_1 = s_2s_1s_2s_1s_2, \quad s_2^4 = s_1s_2s_1s_2s_1$$

which can readily be derived from the given conditions. Hence

$$\begin{aligned}
(s_1s_2^{-1}s_1^{-1}s_2)^2 &= s_1^{-4}(s_1s_2^{-1}s_1s_2)^2 = (s_2s_1s_2^2)^2 = s_2s_1s_2^3s_1s_2^2\\
(s_1s_2^{-1}s_1^{-1}s_2)^3 &= s_1^{-2}s_1s_2^{-1}s_1s_2^2s_1s_2^3s_1s_2^2 = s_2s_1s_2^3s_1s_2^3s_1s_2^2\\
&= s_2^2s_1s_2s_1s_2^5s_1s_2s_1s_2^5s_1s_2s_1s_2^3 = s_2^2s_1s_2^3s_1s_2^3\cdot s_1^8\\
&= s_2^3s_1s_2s_1s_2^5s_1s_2s_1s_2^4\cdot s_1^8 = s_2^3s_1s_2^2s_1s_2^4\cdot s_1^{12}\\
(s_1s_2^{-1}s_1^{-1}s_2)^5 &= s_2s_1s_2^3s_1s_2^2s_2^3s_1s_2^2s_1s_2^4s_1^{12} = s_1^{22}.
\end{aligned}$$

As s_1^{22} is tranformed into its inverse by s_1 it results that the order of s_1 divides 44 and that the order of G divides 1,320. It is easy to see that if two commutative operators satisfy the given conditions they must have 11 for their common order. Hence the following theorem: *If two non-commutative operators satisfy the two conditions* $s_1^2 = s_2^5 = (s_1s_2)^3$ *they generate one of the following four groups: the icosahedral group,* G_{120} *or the direct product of one of these groups and the group of order* 11. *If two commutative operators satisfy these conditions they generate the group of order* 11.

It remains to consider the case when $s_1^2 = s_2^3 = (s_1s_2)^5$, and hence $s_1 = s_2s_1s_2s_1s_2s_1s_2s_1s_2$, $s_2^2 = s_1s_2s_1s_2s_1s_2s_1s_2s_1$. We shall consider again the powers of $s_1s_2^{-1}s_1^{-1}s_2$, and reduce the expressions for these powers by means of the equations which have just been given. Hence the following equations

$$\begin{aligned}
(s_1s_2^{-1}s_1^{-1}s_2)^2 &= s_1^{-4}(s_1s_2^{-1}s_1s_2)^2 = (s_2s_1s_2s_1s_2s_1s_2^2)^2 = s_1^4s_2s_1s_2s_1s_2^2s_1s_2s_1s_2^2\\
(s_1s_2^{-1}s_1^{-1}s_2)^3 &= s_2s_1s_2s_1s_2s_1s_2^2s_1^4s_2s_1s_2s_1s_2\ s_1s_2s_1s_2^2 = s_1^8s_2s_1s_2s_1s_2^2s_1s_2^2s_1s_2s_1s_2^2\\
&= s_1^8s_2s_1s_2s_1s_2^2\cdot s_2s_1s_2s_1s_2s_1s_2s\ s_2\cdot s_2^2s_1s_2s_1s_2^2\\
&= s_1^{16}s_2s_1s_2^2s_1s_2s_1s_2^2s_1s_2^2\\
(s_1s_2^{-1}s_1^{-1}s_2)^5 &= s_1^{20}s_2s_1s_2^2s_1s_2s_1s_2^2s_1s_2^2s_2s_1s_2s_1s_2^2s_1s_2s_1s_2^2 = s_1^{38}.
\end{aligned}$$

* Transactions of the American Mathematical Society, vol. 8 (1907), p. 10.

† American Journal of Mathematics, vol. 20 (1898), p. 277.

As s_1^{38} is transformed into its inverse by s_1 the order of s_1 is a divisor of 76 and the order of G is a divisor of 2,280. If s_1, s_2 are commutative and satisfy the given conditions they must both be of order 19. Hence it is easy to deduce the following theorem: *If two non-commutative operators satisfy the two conditions $s_1^2 = s_2^3 = (s_1 s_2)^5$ they generate one of the following four groups: the icosahedral group, G_{120} or the direct product of one of these groups and the group of order* 19. *If two commutative operators satisfy these conditions they enerate the group of order* 19.

UNIVERSITY OF ILLINOIS.

A TYPE OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION.

By L. A. Howland.

1. The object of this paper is to discuss the homogeneous linear differential equation

$$A(y) = y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} = 0 \tag{1}$$

(the a's being analytic functions of x in a region R and not all constant*) *which has a fundamental system of integrals of the form*

$$y_1, y_1', y_1'', \cdots y_1^{(n-1)}. \tag{2}$$

A necessary condition for this is that there be no relation of the form

$$\sum_{h=0}^{n-1} c_h y_1^{(h)} \equiv 0,$$

where the c's are constants, not all zero; that is, y_1 may not be an integral of an equation of order less than n with constant coefficients.

A further necessary condition is obtained as follows: the equation

$$A'(y) = y^{(n+1)} + \sum_{k=0}^{n-1} a_k y^{(k+1)} + \sum_{k=0}^{n-1} a_k' y^{(k)} = 0$$

has the integrals (2) and, since $y_1', \cdots y_1^{(n-1)}$ are integrals of (1), the equation

$$B_1(y) = \sum_{k=0}^{n-1} a_k' y^{(k)} = 0 \tag{3}$$

has the integrals $y_1, y_1', \cdots y_1^{(n-2)}$. Similarly

$$B_1'(y) = \sum_{0}^{n-1} a_k' y^{(k+1)} + \sum a_k'' y^{(k)} = 0$$

has the same integrals as (3) and hence

$$B_2(y) = \sum_{0}^{n-1} a_k'' y^{(k)} = 0$$

has the integrals $y_1, y_1', \cdots y_1^{(n-3)}$.

Continuing in this way we have finally

* They may be also singled-valued functions of a real variable in an interval I, possessing differential coefficients of all orders up to $n - 1$ inclusive at each point of I.

$$B_{n-1}(y) = \Sigma a_k^{(n-1)} y^{(k)} = 0,$$

which has the integral y_1.

Setting $A(y) = B_0(y)$ we have the system of n equations

$$B_h(y) = 0 \qquad (h = 0, 1, 2 \cdots n-1), \tag{4}$$

which have the common integral y_1.

The converse theorem is also true: *If the system (4) has a common integral y_1, then $y_1, y_1', \cdots y_1^{(n-1)}$ are integrals of $B_0(y) = 0$*; for the following is an identity

$$B_k'(y) \equiv B_k(y') + B_{k+1}(y). \tag{5}$$

By hypothesis we have

$$B_k(y_1) \equiv 0 \qquad (k = 0, 1 \cdots n-1), \tag{6}$$

and from (5) and (6) follows immediately

$$B_k(y_1') \equiv 0 \qquad (k = 0, 1 \cdots n-2),$$

$$B_k(y_1'') \equiv 0 \qquad (k = 0, 1 \cdots n-3),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$B_0(y_1^{(n-1)}) \equiv 0.$$

We have thus the theorem: *The necessary and sufficient condition that the equation* (1) *have a fundamental system of the form* (2) *is that there exist an integral of the system* (4) *which is not an integral of an equation of order less than n with constant coefficients.*

II. We have shown that y_1 is an integral of the system of equations

$$\begin{aligned} y^{(n)} + \sum_0^{n-1} a_k y^{(k)} &= 0, \\ \sum_0^{n-1} a_k' y^{(k)} &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot \\ \sum_0^{n-1} a_k^{(n-1)} y^{(k)} &= 0. \end{aligned} \tag{7}$$

Assume that among the $n + 1$ functions $a_0, a_1, \cdots a_{n-1}, a_n \equiv 1$, there will be a set of n which are linearly independent, *i. e.*, in the matrix

$$M = \begin{vmatrix} 1 & a_{n-1} & \cdots & a_1 & a_0 \\ 0 & a_{n-1}' & \cdots & a_1' & a_0' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_{n-1}^{(n-1)} & \cdots & a_1^{(n-1)} & a_0^{(n-1)} \end{vmatrix}$$

there will be one n-rowed determinant, the Wronskian of this set, which does not vanish identically. That this is always the case is shown below.

We may then solve the equations (7) in the form

$$(8)\quad y : y' : \cdots y^{(n-1)} : y^{(n)} = W_0 : -W_1 : \cdots (-1)^{n-1}W_{n-1} : (-1)^n W_n,^*$$

W_i being the Wronskian of $a_0 \cdots a_{i-1}, a_{i+1} \cdots a_n$.

Now one of the W's is not identically zero, moreover it must be one of the set $W_0 \cdots W_{n-1}$, for because of the constitution of the first column in M, these W's are the first minors of W_n which must vanish with them.†

Suppose W_i is the non-vanishing W and suppose further that W_j does vanish identically (j being one of the numbers $0, 1, \cdots, n-1$). We then have $y^{(i)} : y^{(j)} = W_i : W_j$ or $y^{(j)} = 0$. But we have shown that y may not satisfy an equation of order less than n with constant coefficients. Therefore W_j may not vanish identically.

We have then at once from (8)

$$(9)\quad \begin{aligned} y' &= -\frac{W_1}{W_0}y = -\alpha y, \\ y_1 &= e^{-\int \alpha dx}, \end{aligned}$$

and the equation is solved, the other integrals being obtained by differentiation.

III. Suppose every set of $k+1$ of the functions $a_0, \cdots, a_n$ is linearly dependent but that at least one set of k functions is independent. The matrix M is then of rank k.

Consider the first k equations of (7)

$$(10)\quad \begin{aligned} y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y &= 0, \\ a_{n-1}'y^{(n-1)} + \cdots + a_0'y &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n-1}^{(k-1)}y^{(n-1)} + \cdots + a_0^{(k-1)}y &= 0. \end{aligned}$$

If the $k-1$ functions $a_n \equiv 1, a_{n-1} \cdots a_{n-k+2}$ are linearly dependent, we will add to them such other a's as will form with them a set of $k-1$ independent functions. This set will form a fundamental system of integrals of a homogeneous linear equation of order k:

$$(11)\quad Kz^{(k-1)} + \cdots + Bz' + Az = 0.$$

* Bôcher, Introduction to Higher Algebra, § 17, Theorem 4.

† This means that $a_n = 1$ must be included in the set, or in one of the sets, of n independent functions.

There will be an a, call it $\bar{a}$, which is not a solution of (11); otherwise there could be no linearly independent set of k a's. On the other hand every set of $k+1$ a's is linearly dependent and hence every a may be expressed linearly in terms of $\bar{a}$ and of the $k-1$ a's which form the fundamental integrals of (11), *i. e.*, every a has the form $a_k = c_k\bar{a} + \alpha_k$ (c_k being constant) where α_k is a solution of (11).

We substitute these forms for the coefficients in equations (10), multiply the equations in order by $A, B, \cdots K$, add, and, remembering that the α's are solutions of (11), we obtain:

$$(K\bar{a}^{(k-1)} + \cdots + B\bar{a}' + A\bar{a})(c_{n-1}y^{(n-1)} + \cdots + c_1y' + c_0y) = 0,$$

whence

$$c_{n-1}y^{(n-1)} + \cdots + c_1y' + c_0y = 0.$$

Not all these c's are zero, otherwise every a would be an integral of (11). y then is an integral of an equation of order less than n with constant coefficients. We showed, however, in § 1 that this was impossible, and hence our assumption that M is of rank $k<n$ leads to a contradiction.

IV. We have shown that the equation (1) cannot have integrals of the form (2) unless there is a set of n a's which are linearly independent. We have shown that in case this is true the integral is

$$y_1 = e^{-\int\frac{w_1}{w_0}dx}.$$

We may now obtain conditions which with the above will form a set of conditions necessary and sufficient that the equations $-B_h(y) = 0$ have a common integral y, in case $y, y', \cdots, y^{(n-1)}$ are linearly independent. We need merely to express the fact that $y_1 = e^{-\int\frac{w_1}{w_0}dx}$ satisfies these equations. The conditions will consist of the vanishing of $n-1$ differential expressions in the a's.*

For an equation of second order

$$y'' + a_1y' + a_0y = 0$$

we have the integral

$$y_1 = e^{-\int\frac{a_0'}{a_1'}dx}$$

and the single condition †

* It is obvious that for the equation for which $a_k = c_kx^k$, c^k being constant, these must be the conditions that the equation

$$\sum_1^n [c_km(m-1)\cdots(m-k+1)] + c_0 = 0, \quad (c_n = 1).$$

has n roots forming an arithmetical progression with common difference 1.

† For the same condition obtained somewhat differently cf. Craig, American Journal of Mathematics, No. 8.

$$(a_0')^2 - a_0''a_1' + a_0'a_1'' - a_0'a_1a_1' + a_0(a_1')^2 = 0 \tag{13}$$

For an equation of third order

$$y''' + a_2y'' + a_1y' + a_0y = 0$$

we have the integral

$$y_1 = e^{-\int\frac{a_0''a_2'-a_0'a_2''}{a_1''a_2'-a_1'a_2''}dx} = e^{-\int\alpha dx}$$

and the two conditions

$$3\alpha\alpha' - \alpha^3 - \alpha'' + a_2(\alpha^2 - \alpha') - a_1\alpha + a_0 = 0,$$

and

$$a_2'(\alpha^2 - \alpha') - a_1'\alpha + a_0' = 0,$$

and so on.

The general linear homogeneous equation of second order can always be transformed into an equation of the sort just treated. Let the equation be

$$y'' + p_1y' + p_0y = 0.$$

This goes over by the transformation

$$z = \int\frac{dx}{\varphi(x)}$$

into

$$\frac{d^2y}{dz^2} + (p_1\varphi - \varphi')\frac{dy}{dz} + p_0\varphi^2y = 0.$$

Substituting $a_1 = p_1\varphi - \varphi'$, $a_0 = p_1\varphi^2$ in the equation (13), we obtain a non-linear equation of third order for φ

If φ_1 is an integral of the equation

$$\varphi' = m + p_1\varphi + \frac{p_0}{m}\varphi^2, \tag{14}$$

in other words, if $a_0 = -(m^2 + a_1m)$, m being a constant, the transformed equation

$$\frac{d^2y}{dz^2} + a_1\frac{dy}{dz} - (m^2 + a_1m)y = 0$$

will have an integral satisfying an equation of first order with constant coefficients, viz.: e^{mz}. Conversely, if the transformed equation has an integral of this sort, we must have $a_0 = -(m^2 + a_1m)$ or φ must be an integral of (13).

If, however, φ is any integral of (13) independent of φ_1, the transformed equation will not have an integral e^{mz} but will be of the desired form.

The general equation of nth order cannot be transformed in this way, for φ must in this case be a common integral of $n - 1$ differential equations whose coefficients are differential expressions in the coefficients of the given equation.

Illustrative problem.

$$4x^6y'' + 4x^3(2x^2 + 1)y' + (1 - 2x^2)y = 0.$$

Let $z = \int \frac{dx}{\varphi}$ and the equation becomes

$$\frac{d^2y}{dz^2} + \left[\frac{2x^2 + 1}{x^3}\varphi - \varphi'\right]\frac{dy}{dz} + \varphi^2\frac{1 - 2x^2}{4x^6}y = 0.$$

A solution for φ is seen to be $\varphi = 2x^2$ and as this is not an integral of

$$\varphi' = m + \frac{2x^2 + 1}{x^3}\varphi + \frac{1 - 2x^2}{4mx^6}\varphi^2,$$

the transformed equation

$$\frac{d^2y}{dz^2} - 4z\frac{dy}{dz} + (4z^2 - 2)y = 0$$

is of the type considered. A fundamental system of integrals is

$$y_1 = e^{-\int \frac{a_0'}{a_1'}dz} = e^{\int 2z\,dz} = e^{z^2} \quad \text{and} \quad y_1' = 2ze^{z^2}.$$

The integrals of the given equation are therefore

$$e^{\frac{1}{4x^2}} \quad \text{and} \quad \frac{1}{x}e^{\frac{1}{4x^2}}.$$

The adjoint equation is, of course, also soluble. It is

$$4x^6z'' - 4x^3(2x^2 + 1)z' + (1 + 10x^2 + 8x^4)z = 0$$

with integrals:

$$z_1 = xe^{-\frac{1}{4x^2}} \quad \text{and} \quad z_2 = -x^2e^{-\frac{1}{4x^2}}.$$

The adjoint of the transformed equation is:

$$\frac{d^2\zeta}{dz^2} + 4z\frac{d\zeta}{dz} + (4z^2 + 2)\zeta = 0,$$

with integrals:

$$\zeta_1 = 2ze^{-z^2} \quad \text{and} \quad \zeta_2 = \frac{1}{z}e^{-z^2}.$$

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ON THE COMPLETE LOGARITHMIC SOLUTION OF THE CUBIC EQUATION.

BY R. E. GLEASON.

In this article we shall consider our equation to be in one of the two forms

$$(a)\ x^3 = qx + r \qquad (b)\ x^3 + qx = r$$

in which we may consider both r and q positive without loss of generality since changing the sign of r changes the signs of the roots.

The cubic and quadratic equations arising in practical work are, since their coefficients usually contain decimals, very different from the ideal ones of the classroom. The direct transformation of a complete cubic, all of whose constants are decimal fractions of say four places, by Horner's method into one of the above trinomial cubics is laborious. It is, however, possible easily to effect this transformation logarithmically by the use of tables of logarithms of sums and differences. Consider the general cubic

$$az^3 + bz^2 + cz + d = 0.$$

Setting

$$z = \frac{b}{3a}(x - 1), \quad Q = \frac{9ac}{b^2}, \quad R = \frac{27a^2d}{b^3},$$

we have

$$q = |(Q - 2) - 1|, \quad r = |(Q - 2) - R|,$$

and therefore

$$\log q = \log |(Q - 2) - 1|, \quad \log r = \log |(Q - 2 - R|,$$

considering q to be in (a) or (b) according as $Q - 3$ is negative or positive, and the roots to be $\pm x_1$, $\pm x_2$, $\pm x_3$ according as $Q - R - 2$ is positive or negative. Using a sum or difference table, according to the positive or negative qualities of the quantities involved, we first find $\log |Q - 2|$ and then $\log |(Q - 2) - 1|$ and $\log |(Q - 2) - R|$. It is quicker to find $\log |(Q-2)-1|$ from the sum or difference table after having found $\log |Q-2|$ than it is to find $\log |Q - 3|$, hence the above arrangement appears to be the best. This solution may be justly called purely logarithmic, since only the logarithms of the constants are required throughout the computation, and no other than the ordinary tables are employed.

Corresponding to (a), we have the identities

$$\cos^3 \varphi = \tfrac{3}{4} \cos \varphi + \tfrac{1}{4} \cos 3\varphi$$

$$\cosh^3 \varphi = \tfrac{3}{4} \cosh \varphi + \tfrac{1}{4} \cosh 3\varphi$$

the former to be used when all the roots are real, the latter when there is but one real root.

Corresponding to (*b*), which has but one real root, we have

$$\sinh^3 \varphi + \tfrac{3}{4} \sinh \varphi = \tfrac{1}{4} \sinh 3\, \varphi.$$

A single formula suffices for the determination of the imaginary roots of both (*a*) and (*b*). Let the imaginary root of either (*a*) or (*b*) be $-\xi \pm \eta \sqrt{-1} = \mu (\cos \theta \pm \sqrt{-1} \sin \theta)$. If x_1 is the real root, we then have $\xi = \frac{1}{2} x_1$ and $r = x_1 \mu^2$, or $\mu = \sqrt{(r/x_1)}$. Therefore

$$\cos \theta = \sqrt{\frac{x_1^3}{4r}}, \quad \eta = \sin \theta \cdot \sqrt{\frac{r}{x_1}}.$$

This is a much simpler trigonometric solution for the complex roots than that given by Chauvenet,* which is not only involved but also requires separate formulæ for (*a*) and (*b*). The latter solution is given for comparison.

For (*a*) put

$$\sin \theta = \sqrt{\frac{4q^3}{27r^2}}, \quad \tan \tfrac{1}{2}\varphi = \sqrt[3]{\tan \tfrac{1}{2}\theta},$$

then

$$\eta = \tfrac{1}{2} x_1 \sqrt{3} \cdot \cos \varphi.$$

For (*b*) put

$$\tan \theta = \sqrt{\frac{4q^3}{27r^2}}, \quad \tan \tfrac{1}{2}\varphi = \sqrt[3]{\tan \tfrac{1}{2}\theta},$$

then

$$\eta = \tfrac{1}{2} x_1 \sqrt{3} \cdot \sec \varphi.$$

The former trigonometric solution can be generalized into a logarithmic method of computing the two remaining roots (either conjugate complex or real numbers) when $n - 2$ roots of an equation of the nth degree are known, circular functions being used when the remaining pair are imaginary and hyperbolic functions when they are real. Thus for computing the complex roots $-\xi \pm \eta \sqrt{-1}$ of

$$x^n + p_1 x^{n-1} + \cdots + p_n = 0$$

when $n - 2$ of its roots are known, we have

$$\xi = \tfrac{1}{2}(p_1 + x_1 + \cdots + x_{n-2}), \quad \mu = \sqrt{\left| \frac{p_n}{x_1 \cdots x_{n-2}} \right|},$$

$$\cos \theta = \left| \frac{\xi}{\mu} \right|, \quad \eta = \mu \sin \theta.$$

* Plane and Spherical Trigonometry, pp. 97, 98.

If the remaining pair are real and are represented by $-\xi \pm \eta$, where and μ have their previous values we have

$$\cosh \theta = \left|\frac{\xi}{\mu}\right|, \quad \eta = \mu \sinh \theta.$$

This is particularly useful as a logarithmic solution of the quadratic equation, where more often than not in practical work the coefficients contain decimal fractions, and the algebraic formulæ are consequently rendered cumbersome for computation.

Substituting the numerical values of the logarithms of the constants, the above can be arranged in the following table.

Table for the Complete Logarithmic Solution of the Cubic Equation.

$$(a)\ x^3 = qx + r, \quad (b)\ x^3 + qx = r.$$

q and r are both positive. Changing the sign of r changes the signs of the roots. Denote the complex roots by $-\xi \pm \eta\sqrt{-1}$.

I. Case (a). $4q^3 \not< 27r^2$. Three real roots x_1, x_2, x_3,

$$\log \cos 3\varphi = 0.41465 + \log r - \tfrac{3}{2} \log q, \tag{1}$$

$$\log x_1 = 0.06247 + \tfrac{1}{2} \log q + \log \cos \tfrac{1}{3}\theta, \tag{2}$$

$$\log(-x_2) = 0.06247 + \tfrac{1}{2} \log q + \log \cos \tfrac{1}{3}(\pi - \theta),$$

$$\log(-x_3) = 0.06247 + \tfrac{1}{2} \log q + \log \cos \tfrac{1}{3}(\pi + \theta),$$

where θ is the smallest positive value of 3φ satisfying (1).

II. Case (a). $4q^3 < 27r^2$. One real root x_1.

Replace cos by cosh in (1) and (2). $\log \eta = \tfrac{1}{2} \log q + \log \sinh \tfrac{1}{3}\vartheta$ *

III. Case (b). One real root x_1.

Replace cos by sinh in (1) and (2). $\log \eta = \tfrac{1}{2} \log q + \log \cosh \tfrac{1}{3}\vartheta$ *

IV. To calculate the imaginary roots in II and III.

Let them be $-\xi \pm \eta\sqrt{-1} = \mu(\cos\theta \pm \sqrt{-1}\sin\theta)$; then

$$\xi = \tfrac{1}{2}x_1,$$

$$\log \cos \theta = \tfrac{3}{2} \log x_1 + 9.69897 - \tfrac{1}{2} \log r - 10,$$

$$\log \eta = \tfrac{1}{2}(\log r - \log x_1) + \log \sin \theta.$$

Palo Alto, California.

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* Corresponding to $\cos\frac{1}{3}(\pi \pm \vartheta)$ we have $\cosh\frac{1}{3}(\pi\sqrt{-1} \pm \vartheta)$ or $\sinh\frac{1}{3}(\pi\sqrt{-1} \pm \vartheta)$ which when expanded give above values of η.

THE CIRCULAR NUMBERS FOR A PLANE CURVE.

By Horace T. Burgess.*

Introduction.—The well-known numbers of Plücker for a plane curve are arithmetical invariants of the curve under the transformations of the projective group in the plane. The object of this paper is to derive a set of numbers for a plane curve such that each is an arithmetical invariant of the curve under the transformations † of the circular group in the plane; to establish the equations connecting these numbers; and to illustrate by some simple examples.

1. **Derivation.**—Let γ be a plane curve; project γ stereographically upon a sphere Σ and denote the projection by Γ. Any projective transformation of space leaving Σ invariant is equivalent to a transformation of the circular group in the plane of γ.‡ Any numbers which are arithmetical invariants of Γ under the projective group G_6, which leaves Σ invariant, must be invariants of γ under the circular G_6 in the plane of γ. We shall choose as our invariants of Γ under the projective G_6 Cayley's numbers§ $m, r, n, \alpha, \beta, x, y, g, h$, together with the five numbers d, i, j, p, f, defined as follows: We shall take d to be the number of actual double points upon Γ; if we refer to the two systems of generators upon Σ as the i-system and the j-system, then i and j are the number of points of Γ in common with each generator of the i-system and j-system respectively; and p and f, we define by the invariant equations $2p = i(i-1) + j(j-1)$ and $f = (r-2i)(r-2j)$. We have also $m = i + j$.‖

Let O be the center of projection on Σ. If γ meets the line at infinity, IJ, only in the circular points I and J, Γ does not pass through O. Excluding the circular points I and J: If γ intersects IJ in t distinct points, Γ has t distinct branches through O; if γ touches IJ, Γ has a cusp at O; if γ has contact of higher order with IJ, Γ has a singularity of higher order at O. A singularity arising at O on Γ in this manner is equivalent to a certain number of nodes and cusps,** and these in general may be included

* Presented to the American Mathematical Society (Minneapolis, 1910).

† Newson, Bulletin of the American Mathematical Society, Dec., 1897.

‡ Klein, Einleitung in die höhere Geometrie, I, pp. 378–380.

§ Salmon, Solid Geometry, 4th ed., pp. 291–5.

‖ Clebsch-Lindemann, Vorlesungen über Geometrie, Band II, p. 418.

** Charlotte Scott, American Journal of Mathematics, Vol. 14, pp. 301–325. The singularity arising on Γ at O due to a given order of contact of γ with IJ is of the same type as that arising in quadric inversion in the plane with OIJ as the fundamental triangle.

in the numbers d and β for Γ such that Cayley's equations* are satisfied. Those plane curves, which on projection give rise to singularities at O so complicated that Cayley's equations are not satisfied in the above manner, must be excluded from our consideration. Since Γ lies entirely upon Σ, each one of Cayley's numbers for Γ has a particular significance. This is so obvious geometrically that we shall state it briefly in the following parallel columns which are self-explanatory:

Cayley's Numbers for Γ.	*Cayley's Numbers for* Γ *upon* Σ.
m: The degree of Γ is the number of points of Γ in an arbitrary plane E.	The number of intersections of Γ with the circle cut out of Σ by E.
r: The rank of Γ is the number of tangents of Γ cutting an arbitrary line L.	The number of circles, cut out of Σ by the pencil of planes through L, which touch Γ.
n: The class of Γ is the number of osculating planes of Γ through an arbitrary point P.	The number of osculating circles of Γ cut out of Σ by the bundle of planes through P. When P is upon Σ, the circles all pass through P.
α: The number of stationary planes of Γ.	The number of stationary circles of Γ.
β: The number of cusps of Γ.	The number of cusps of Γ.
d: The number of nodes of Γ.	The number of nodes of Γ.
x: The number of points on two tangents of Γ which lie in an arbitrary plane E.	The number of doubly orthogonal circles of Γ cut out of Σ by the bundle of planes through the pole of E with respect to Σ. When E is tangent to Σ at the point P, the circles all pass through P.
y: The number of planes through two tangents of Γ which also pass through an arbitrary point P.	The number of bitangent circles of Γ cut out of Σ by the bundle of planes through P. When P is upon Σ, the circles all pass through P.
g: The number of lines in two osculating planes of Γ which lie in an arbitrary plane E.	The number of pairs of osculating circles of Γ which intersect on the circle cut out of Σ by E.

* Cf. Salmon, loc. cit., p. 295.

h: The number of lines through two points of Γ and an arbitrary point P. This includes both the actual and apparent double points of Γ.	The number of pairs of tangent circles of Γ cut out of Σ by the planes through P such that each circle passes through the point of contact of the other.

p: It will be shown later that p is the number of apparent double points of Γ upon Σ.

f: If r is the rank of Γ, then any line L, which intersects Γ in κ distinct points, is met by $r - 2\kappa$ tangents of Γ excluding the κ tangents at the κ points common to Γ and L. Further if L is a generator of Σ belonging to the i-system, it meets Γ in i points, and L is met by $r - 2i$ tangents of Γ which are generators of the j-system; for all of the tangents of Γ which meet a generator of Σ are generators of Σ. Hence any generator of the i-system is met by $r - 2i$ generators of the j-system which are tangent to Γ; and any generator of the j-system is met by $r - 2j$ generators of the i-system which are tangent to Γ. These $r - 2i$ generators of the j-system and $r - 2j$ generators of the i-system meet in f points.

To get the geometrical significance of these invariants of γ, we project Γ back stereographically into γ together with the configurations we have just stated. We shall use the corresponding capital letters to denote these projected configurations in the plane. Singularities on Γ which are not at the center of projection O project into the same singularities on γ. If Γ has a singularity at O, on projection it breaks up into intersections and contacts of various orders with the line IJ; and it follows that B and D corresponding respectively to β and d are not the number of nodes and cusps on γ. B we shall call the *cuspidal equivalence of* γ; and in like manner, excluding the nodes at I and J, D we shall call the *nodal equivalence of* γ. The generators of Σ project into the minimum lines of the plan; and if the i-system projects into the I-pencil, the j-system projects into the J-pencil. The projecting lines OI and OJ are generators of Σ; OI belongs to the j-system and OJ belongs to the i-system; the f points, in which the $r - 2i$ generators tangent to Γ and meeting OJ, and the $r - 2j$ generators tangent to Γ and meeting OI intersect, project into the foci of γ.

The significance of the other projected configurations seems to be so obvious that we state without further consideration the following:

THEOREM I. *The plane curve γ has the following invariant numbers and geometrical configurations under the transformations of the circular group G_6:*

M: *The circular degree of γ is the number of points of γ in common with an arbitrary circle excluding the circular points I and J.*

R: *The circular class of γ is the number of circles of an arbitrary pencil touching γ.*

N: *The circular inflexion of γ is the number of osculating circles of γ through an arbitrary point.*

A: *The number of stationary circles of γ.*

B: *The cuspidal equivalence of γ.*

D: *The nodal equivalence of γ excluding the nodes at I and J.*

X: *The number of doubly orthogonal circles of γ through an arbitrary point.*

Y: *The number of bitangent circles of γ through an arbitrary point.*

G: *The number of pairs of osculating circles of γ intersecting upon an arbitrary circle.*

H: *The number of pairs of tangent circles of γ cutting an arbitrary circle orthogonally while each circle of a pair passes through the point of contact of the other.*

I: *The number of points of γ in common with a minimum line through I, excluding I.*

J: *The number of points of γ in common with a minimum line through J, excluding J.*

F: *The number of foci of γ.*

2. **Circular Curves.**—Those plane curves which meet the line IJ only in the circular points, we call *circular curves.* If γ is a *circular* curve, Γ does not pass through O and the cone which projects Γ into γ shows $M = \mu$, $R = \nu$, $N = i$, $B = \kappa$, $Y = \tau$, $H = \delta$, where μ, ν, etc., are Plücker's numbers for γ.* Further, in our definition of x, if we choose the arbitrary plane E tangent to Σ at O, all the doubly orthogonal circles pass through O and each projects into a binormal of γ. Likewise in our definition of g, if we take the arbitrary plane E through O, we have, on projecting, the number of pairs of osculating circles of γ having a common radical axis. P is the number of nodes at I and J; for γ has J branches through I and I branches through J.

These considerations enable us to state the following theorem for *circular* curves in the plane:

THEOREM II. *If γ is a circular curve, it has the following geometrical properties:*

$M = \mu$, *the Plücker degree.*
$R = \nu$, *the Plücker class.*
$N = i$, *the number of inflexions.*
$A =$ *the number of stationary circles.*
$B = \kappa$, *the number of cusps.*
$D =$ *the number of nodes, excluding those at I and J.*
$X =$ *the number of binormals.*
$Y = \tau$, *the number of bitangents.*

* Cf. Salmon, loc. cit., p. 295.

G = *the number of pairs of osculating circles having a common radical axis.*
H = δ, *the number of nodes.*
I = *the number of branches through* J.
J = *the number of branches through* I.
P = *the number of nodes at* I *and* J.
F = *the number of foci.*

It is obvious that any plane curve may be transformed into a *circular* curve by a transformation of the circular group G_6; for we have only to transform Γ by a rotation of Σ such that the transformed curve on Σ does not pass through O and the equivalent transformation of the circular group transforms γ into a *circular* curve. We have then from Theorem II the particular significance of these numbers for the *circular* curve into which any *noncircular* curve may be transformed by a transformation of the circular group G_6.

3. **Equations.**—The circular numbers satisfy the following equations:

$$M = 3N(N-2) - 6G - 8A, \tag{1}$$

$$M = R(R-1) - 2Y - 3N, \tag{2}$$

$$R = N(N-1) - 2G - 3A, \tag{3}$$

$$R = M(M-1) - 2H - 3B, \tag{4}$$

$$N = R(R-1) - 2X - 3M, \tag{5}$$

$$N = 3M(M-2) - 6H - 8B, \tag{6}$$

$$A = 3R(R-2) - 6X - 8M, \tag{7}$$

$$B = 3R(R-2) - 6Y - 8N, \tag{8}$$

$$P = \tfrac{1}{2}[I(I-1) + J(J-1)], \tag{9}$$

$$F = (R-2I)(R-2J), \tag{10}$$

$$M = I + J, \tag{11}$$

$$H = D + P. \tag{12}$$

The first eight of these equations are taken directly from those of Cayley for Γ; hence, when any three of the circular numbers M, R, N, A, B, X, Y, G, H are known, the remaining six may be found. The last equation follows directly from Theorem II, and shows p to be the number of apparent double points of Γ on Σ. The other three equations follow from those already stated for space.

4. **Equivalence.** Definition: *Two plane curves are equivalent with*

respect to the circular group provided they are transforms of one another by transformations of this group.

THEOREM III. *A necessary condition for the equivalence of two plane curves with respect to the circular group is that they have the same circular numbers.*

The truth of this theorem follows directly from the above definition and the invariance of the circular numbers. That this theorem is not sufficient will be shown by one of the following examples.

5. **Examples.**—In general, to find the circular numbers for a plane curve γ, we find B, D, I, and J from which we calculate the others by means of our equations. In this manner we find the characteristics of the familiar curves as given in the following table:

Curve	M	R	N	A	B	D	X	Y	G	H	I	J	P	F
Parabola	4	5	4	1	1	0	2	2	2	2	2	2	2	1
Ellipse *	4	6	6	4	0	1	6	4	6	3	2	2	2	4
Witch	6	10	12	12	0	4	30	24	43	10	3	3	6	16
Cassini's oval	4	8	12	16	0	0	16	8	38	2	2	2	2	16
$y^2 = ax^3$	6	7	6	3	3	1	9	9	7	7	3	3	6	1

The parabola, cissoid, and cardioid are equivalent† and hence have the same circular numbers by Theorem III. The ellipse, hyperbola, lemniscate, limaçon, and strophoid are found, in the above-mentioned manner, to possess the same circular numbers; likewise the witch, folium, and three-leaved rose are found to have the same circular numbers. The ellipse and hyperbola are not equivalent for there exists no transformation of our G_6 which will transform the one into the other; this example shows that Theorem III is not sufficient. The ellipse and the limaçon with the conjugate point are equivalent;‡ the hyperbola and the limaçon with the node are equivalent.‡ The equilateral hyperbola, lemniscate, and strophoid are equivalent.§

The curve $y = x^3$ furnishes the simplest example of a case where the circular numbers do not satisfy our equations. This curve has a cusp on the line at infinity.

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* Cf. Klein, Math. Annalen, Vol. 10, p. 208.

† Cf. Smith and Gale, Analytic Geometry, pp. 299, 302.

‡ Cf. Smith and Gale, loc. cit., p. 302.

§ Cf. Smith and Gale, loc. cit., pp. 300, 301.

ON THE SUM OF A CERTAIN TRIPLE SERIES.

BY ERNEST W. BROWN.

In obtaining the value of his infinite determinant,* Dr. G. W. Hill finds it necessary to obtain the sum of a series which is equivalent to

$$f(\alpha) = \Sigma_i \Sigma_k \Sigma_{k'} \{i\}\{i+1\}\{i+k\}\{i+k+1\}\{i+k+k'\}\{i+k+k'+1\},$$

where

$$\{j\} = \frac{1}{\alpha^2 - j^2}, \quad i = 0, \pm 1, \pm 2, \cdots; k, k' = 2, 3, \cdots,$$

α being a definite number not a positive or negative integer or zero. In his paper Hill develops the cases in which there are four factors with a simple or double summation and, with reference to $f(\alpha)$, remarks that it may be treated in an analogous manner and then gives its value. If we attempt to follow out Hill's method of partial fractions directly, the algebraic work becomes very heavy and, as some unsuccessful attempts by others appear to have been made in order to sum this particular series, it seems worth while to give in detail the method by which, after many trials, I have succeeded in showing that his expression for it is correct.

In the following developments Σ refers always to the complete sum of the expression to which it is attached unless one of the three letters i, k, k' is inserted, in which case it refers to the particular summation only.

It is to be remarked first that the series $\Sigma\{j\}$, $\Sigma\{j\}\{j'\}$, $\Sigma\{j\}\{j'\}\{j''\}$ are convergent and therefore remain so when each term is multiplied by a finite factor. Hence $f(\alpha)$ is convergent.

In all cases, since i has all integral values between $+\infty$ and $-\infty$, we can replace i by $i \pm$ any integer without altering the sum. Also, since k, k' have the same range, they may be interchanged. Finally,

$$\Sigma_k\{i+k+1\} = \Sigma_k\{i+k\} - \{i+2\}, \text{ etc.}$$

Put $-i-k-1$ for i. Then

$$f(\alpha) = \Sigma\{i\}\{i+1\}\{i+k\}\{i+k+1\}\{i-k'\}\{i-k'+1\}.$$

We have, by partial fractions,

* On a part of the motion of the lunar perigee, etc., Cambridge, 1877 and Acta Math., vol. 8 (1886), pp. 1–36.

$$\{i+k\}\{i+k+1\} = \frac{1}{2\alpha}\cdot\frac{1}{2\alpha-1}\left(\frac{1}{\alpha-i-k-1}+\frac{1}{\alpha+i+k}\right)$$
$$-\frac{1}{2\alpha}\cdot\frac{1}{2\alpha+1}\left(\frac{1}{\alpha-i-k}+\frac{1}{\alpha+i+k+1}\right)$$
$$=\frac{1}{2\alpha}\cdot\left(\frac{1}{2\alpha-1}-\frac{1}{2\alpha+1}\right)\left(\frac{1}{\alpha-i-k}+\frac{1}{\alpha+i+k}\right)$$
$$+\frac{1}{2\alpha}\cdot\frac{1}{2\alpha-1}\left(\frac{1}{\alpha-i-k-1}-\frac{1}{\alpha-i-k}\right)$$
$$-\frac{1}{2\alpha}\cdot\frac{1}{2\alpha+1}\left(\frac{1}{\alpha+i+k+1}-\frac{1}{\alpha+i+k}\right).$$

Every term is finite and therefore $f(\alpha)$, with these two factors so expressed, is convergent. The first line of the latter expression is $2\{i+k\}/(4\alpha^2-1)$. As no other part of $f(\alpha)$ contains k, we can sum the two terms in the last line from $k=2$ to $k=\infty$; the parts of these within the brackets give $-1/(\alpha-i-2)$ and $-1/(\alpha+i+2)$, respectively. Hence

$$\Sigma_k\{i+k\}\{i+k+1\} = \frac{1}{4\alpha^2-1}[2\Sigma_k\{i+k\}-(2i+5)\{i+2\}]$$

(1)

$$=\frac{1}{4\alpha^2-1}[2\Sigma_k\{i+k+1\}-(2i+3)\{i+2\}].$$

Similarly

(2) $$\Sigma_{k'}\{i-k'\}\{i-k'+1\} = \frac{1}{4\alpha^2-1}[2\Sigma_{k'}\{i-k'\}+(2i-1)\{i-1\}].$$

We have further,

$$-(2i+3)\{i+2\}\{i+1\} = \{i+1\}-\{i+2\},$$
$$(2i-1)\{i\}\{i-1\} = \{i\}-\{i-1\}.$$

Using these results in the product of (1) multiplied by $\{i+1\}$ and (2) multiplied by $\{i\}$, we obtain

$$f(\alpha) = \frac{1}{(4\alpha^2-1)^2}\Sigma[2\{i+1\}\{i+k+1\}+\{i+1\}-\{i+2\}]$$
$$\times[2\{i\}\{i-k'\}+\{i\}-\{i-1\}]$$
$$=\frac{1}{(4\alpha^2-1)^2}(4X+4B-4C+d_1-2d_2+d_3),$$

where

$$X = \Sigma\{i\}\{i+1\}\{i+k+1\}\{i-k'\},$$

$$B = \tfrac{1}{2}\Sigma\{i\}\{i+1\}[\{i+k+1\}+\{i-k'\}],$$

$$\begin{aligned} C &= \tfrac{1}{2}\Sigma\{i+1\}\{i-1\}\{i+k+1\}+\tfrac{1}{2}\Sigma\{i+2\}\{i\}\{i-k'\} \\ &= \tfrac{1}{2}\Sigma\{i\}\{i+2\}[\{i+k+2\}+\{i-k'\}], \\ &= \Sigma\{i\}\{i+2\}\{i-k'\} = \Sigma\{i\}\{i-2\}\{i+k\}, \end{aligned}$$

by putting $-i-2$ for i and k' for k, in the former term, and

$$d_1 = \Sigma\{i\}\{i+1\}, \quad d_3 = \Sigma\{i-1\}\{i+2\} = \Sigma\{i\}\{i+3),$$

$$d_2 = \tfrac{1}{2}\Sigma\{i\}\{i+2\}+\tfrac{1}{2}\Sigma\{i-1\}\{i+1\} = \Sigma\{i\}\{i+2\}.$$

We have now to reduce X to a form in which there are three factors. Proceeding as before by partial fractions and separating into three parts, we have

$$\begin{aligned} X &= \frac{1}{2\alpha}\Sigma\left(\frac{1}{2\alpha-1}-\frac{1}{2\alpha+1}\right)\left(\frac{1}{\alpha-i}+\frac{1}{\alpha+i}\right)\{i+k+1\}\{i-k'\} \\ &\quad +\frac{1}{2\alpha}\Sigma\left[\frac{1}{2\alpha-1}\left(\frac{1}{\alpha-i-1}-\frac{1}{\alpha-i}\right)-\frac{1}{2\alpha+1}\left(\frac{1}{\alpha+i+1}-\frac{1}{\alpha+i}\right)\right] \\ &\qquad \{i+k+1\}\{i-k'\}. \end{aligned} \tag{3}$$

The first line of this gives $(2Y-2C)/(4\alpha^2-1)$ where

$$Y = \Sigma\{i\}\{i+k+1\}\{i-k'\}+C = \Sigma\{i\}\{i+k\}\{i-k'\}.$$

For the second line, we note that the double series

$$\Sigma_i\Sigma_{k'}\frac{\{i+k+1\}\{i-k'\}}{\alpha-i-1} = \Sigma_i\Sigma_{k'}\left[\frac{\{i+k\}\{i-k'\}}{\alpha-i}-\frac{\{i+k\}\{i-2\}}{\alpha-i}\right]$$

are convergent; the second form is obtained by putting $i-1$ for i and then $\Sigma_{k'}\{i-k'-1\} = \Sigma_{k'}\{i-k'\}-\{i-2\}$. Hence

$$\begin{aligned} &\Sigma_i\Sigma_{k'}\left(\frac{1}{a-i-1}-\frac{1}{\alpha-i}\right)\{i+k+1\}\{i-k'\} \\ &\quad = \Sigma_i\Sigma_{k'}\left[\frac{\{i+k\}-\{i+k+1\}}{\alpha-i}\{i-k'\}\right]-\Sigma_i\frac{\{i+k\}\{i-2\}}{\alpha-i}. \end{aligned} \tag{4}$$

If now we sum for values of k, the left hand member is convergent and the first term of the right hand member gives a series whose typical term is $\phi(k)-\phi(k+1)$, where $\phi(k)$ is a rational fractional function of k of one

degree less in the numerator than in the denominator. The series is therefore convergent and has a sum $\phi(2)$. Hence (4) is equal to

$$\Sigma_i\Sigma_{k'}\frac{\{i+2\}\{i-k'\}}{\alpha-i}-\Sigma_i\Sigma_k\frac{\{i-2\}\{i+k\}}{\alpha-i}=-2\Sigma i\{i\}\{i-2\}\{i+k\},$$

by putting $-i$ for i, k' for k in the first term.

This gives the first half of the second line of (3); the second half is obtained from it by changing the sign of α. Adding the two portions together we obtain

$$X=\frac{2}{4\alpha^2-1}(Y-C)-\Sigma\frac{4i}{4\alpha^2-1}\{i\}\{i-2\}\{i+k\}.$$

But

$$-4i=(\alpha^2-i^2)-[\alpha^2-(i-2)^2]-4.$$

Hence,

$$X=\frac{1}{4\alpha^2-1}(2Y-2C+\Sigma\{i-2\}\{i+k\}-\Sigma\{i\}\{i+k\}-4C)$$

$$=\frac{1}{4\alpha^2-1}(2Y-6C+\Sigma\{i\}\{i+k+2\}-\Sigma\{i\}\{i+k\})$$

$$=\frac{1}{4\alpha^2-1}(2Y-6C-d_2-d_3).$$

Again

$$Y=\Sigma\{i\}[\{i+k-1\}-\{i+1\}][\{i-k'+1\}-\{i-1\}]$$

$$=Z-2A+d_4,$$

where

$$Z=\Sigma\{i\}\{i+k-1\}\{i-k'+1\},$$

$$A=\tfrac{1}{2}\Sigma[\{i\}\{i+1\}\{i-k'+1\}+\{i\}\{i-1\}\{i+k-1\}]$$

$$=\tfrac{1}{2}\Sigma\{i\}\{i-1\}[\{i-k\}+\{i+k-1\}],$$

by putting $i-1$ for i and k for k' in the first term, and

$$d_4=\Sigma\{i\}\{i-1\}\{i+1\}.$$

Reduction of A, B, C.—Suppose we desire to sum the expression denoted by A with respect to k. We note that the portion in square brackets contains all the functions $\{i+j\}$ where j goes from $-\infty$ to $+\infty$, except those for $j=-1, 0$, that is, all the functions $\{j\}$ except $\{i-1\}$, $\{i\}$. Hence

$$2A=\Sigma_i\Sigma_j\{i\}\{i-1\}\{j\}-\Sigma\{i\}\{i-1\}^2-\Sigma\{i\}^2\{i-1\}.$$

But $\Sigma_j\{j\}=\pi\cot\pi\alpha/\alpha$. Therefore, putting $i+1$ for i in the last term and changing the sign of i throughout, we obtain

$$A = \frac{\pi \cot \pi\alpha}{2\alpha} d_1 - d_5,$$

where

$$d_5 = \Sigma\{i\}\{i+1\}^2.$$

Similarly, putting k for k',

$$2B = \Sigma\{i\}\{i+1\}\{j\} - 2\{i\}\{i+1\}\{i-1\} - 2\{i\}\{i+1\}^2,$$

$$B = \frac{\pi \cot \pi\alpha}{2\alpha} d_1 - d_4 - d_5.$$

Also, with the second form for C, after putting k' for k,

$$C = \frac{\pi \cot \pi\alpha}{2\alpha} d_2 - d_6 - d_7 - \tfrac{1}{2} d_4,$$

where

$$d_6 = \Sigma\{i\}\{i+2\}\{i-1\}, \quad d_7 = \Sigma\{i\}\{i+2\}^2.$$

Gathering together the results so far obtained, we find

$$f(\alpha) = \frac{4}{(4\alpha^2-1)^3}(2Z - 4A - 6C - d_2 - d_3 + 2d_4) + \frac{1}{(4\alpha^2-1)^2}(4B - 4C + d_1 - 2d_2 + d_3)$$

$$(5) \quad = \frac{4}{(4\alpha^2-1)^3}\left[2Z - \frac{\pi \cot \pi\alpha}{\alpha}(2d_1 + 3d_2) - d_2 - d_3 + 5d_4 + 4d_5 + 6d_6 + 6d_7\right]$$
$$+ \frac{1}{(4\alpha^2-1)^2}\left[\frac{2\pi \cot \pi\alpha}{\alpha}(d_1 - d_2) + d_1 - 2d_2 + d_3 - 2d_4 - 4d_5 + 4d_6 + 4d_7\right].$$

All that remains now is to find the sums of the various series $d_1, d_2, \cdots$ with respect to i, and that of Z with respect to i, k, k'.

Value of Z.—Let $p, q, r, \cdots$ be any quantities and let S_1p be their sum, S_2pq the sum of their products taken two at a time, S_3pqr the sum of their products taken three at a time. We have

$$(S_1p)^3 = S_1p^3 + 3S_2p^2q + 6S_3pqr$$
$$= S_1p^3 + 3(S_1p^2)(S_1p) - 3S_1p^3 + 6S_3pqr,$$

therefore

$$S_3pqr = \tfrac{1}{6}S_1p^3 - \tfrac{1}{2}(S_1p^2)(S_1p) + \tfrac{1}{3}S_1p^3.$$

In order to apply this to the computation of Z we note that $i + k - 1$ represents, for integral values of k from 2 to ∞, all integers greater than i, and $i - k' + 1$, for a similar range of k', all integers less than i. Since i has the range $+\infty$ to $-\infty$, Z is the sum of the products, taken three at a time, of the functions $\{i\}$. We therefore have

$$Z = \tfrac{1}{6}(\Sigma\{i\})^3 - \tfrac{1}{2}(\Sigma\{i\}^2)(\Sigma\{i\}) + \tfrac{1}{3}\Sigma\{i\}^3.$$

Now

$$\Sigma\{i\} = \Sigma\frac{1}{\alpha^2 - i^2} = \frac{\pi \cot \pi\alpha}{\alpha}.$$

Differentiating this twice with respect to α, we obtain

$$\Sigma\{i\}^2 = \frac{\pi^2}{2\alpha^2}\cdot\frac{1}{\sin^2 \pi\alpha} + \frac{\pi \cot \pi\alpha}{2\alpha^3},$$

$$\Sigma\{i\}^3 = \frac{\pi^3 \cot \pi\alpha}{4\alpha^3 \sin^2 \pi\alpha} + \frac{3}{8}\frac{\pi^2}{\alpha^4 \sin^2 \pi\alpha} + \frac{3}{8}\frac{\pi \cot \pi\alpha}{\alpha^5},$$

whence

$$Z = \frac{\pi \cot \pi\alpha}{8\alpha^5} - \frac{\pi^2(2\cos^2 \pi\alpha - 1)}{8\alpha^4 \sin^2 \pi\alpha} - \frac{1}{6}\frac{\pi^3 \cot \pi\alpha}{\alpha^3}$$

$$= \frac{\pi \cot \pi\alpha}{\alpha}\left[\frac{1}{8\alpha^4} - \frac{\pi \cot 2\pi\alpha}{4\alpha^3} - \frac{\pi^2}{6\alpha^2}\right].$$

Values of d_1, d_2, d_3.—We need $\{i\}\{i+k\}$ where k is a given integer. We have

$$\Sigma_i\{i\}\{i+k\} = \Sigma_i\frac{4}{(2\alpha+k)^2 - (2i+k)^2}\cdot\frac{4}{(2\alpha-k)^2 - (2i+k)^2}$$

$$= \frac{2}{\alpha k}\Sigma_i\left[\frac{1}{(2\alpha-k)^2 - (2i+k)^2} - \frac{1}{(2\alpha+k)^2 - (2i+k)^2}\right]$$

$$= \frac{\pi \cot \pi\alpha}{\alpha k(2\alpha - k)} - \frac{\pi \cot \pi\alpha}{\alpha k(2\alpha + k)} = \frac{2\pi \cot \pi\alpha}{\alpha(4\alpha^2 - k^2)}.$$

Hence

$$(4\alpha^2 - 1)\Sigma_i\{i\}\{i+k\} = \left(1 + \frac{k^2 - 1}{4\alpha^2 - k^2}\right)\frac{2\pi \cot \pi\alpha}{\alpha}.$$

Values of d_4, d_6.—We have, if $k \neq k'$, by a similar proceedure,

$$\{i\}\{i+k\}\{i+k'\} = \frac{1}{2\alpha(k-k')}\left[\frac{1}{\alpha^2 - i^2}\cdot\frac{4}{(2\alpha - k + k')^2 - (2i + k + k')^2} - \frac{1}{\alpha^2 - i^2}\cdot\frac{4}{(2\alpha + k - k')^2 - (2i + k + k')^2}\right]. \tag{6}$$

By partial fractions, the second term within the square brackets is equal to

$$\frac{A}{\alpha + i} + \frac{B}{\alpha - i} + \frac{C}{\alpha + i + k} + \frac{D}{\alpha - i - k'},$$

where

$$A = -\frac{1}{2\alpha k}\cdot\frac{1}{2\alpha - k'}, \qquad B = \frac{1}{2\alpha k'}\cdot\frac{1}{2\alpha + k},$$

$$C = \frac{1}{k(2\alpha + k)} \cdot \frac{1}{2\alpha + k - k'}, \quad D = -\frac{1}{k'(2\alpha - k')} \cdot \frac{1}{2\alpha + k - k'}.$$

If we add the corresponding term obtained by changing the sign of i we find convergent series. We can therefore write

$$\Sigma_i \frac{1}{\alpha + i + \lambda} = \pi \cot \pi\alpha,$$

where λ is an integer, and the second term of (6) is equal to $(A + B + C + D)\pi \cot \pi\alpha$. The first term of (6) within the square brackets is the same quantity with the sign of α and the sign changed. Adding, we find at once,

$$\Sigma_i \{i\}\{i + k\}\{i + k'\} = \left[\frac{1}{k'(k - k')} \cdot \frac{1}{4\alpha^2 - k^2} - \frac{1}{k(k - k')} \cdot \frac{1}{4\alpha^2 - k'^2} \right.$$
$$\left. - \frac{1}{kk'} \cdot \frac{1}{4\alpha^2 - (k - k')^2}\right] \frac{2\pi \cot \pi\alpha}{\alpha}.$$

Values of d_5, d_7.—We have, by partial fractions,

$$\{i\}\{i + k\}^2 = \frac{A}{\alpha + i} + \frac{B}{\alpha - i} + \frac{C_1}{\alpha + i + k} + \frac{D_1}{\alpha - i - k} + \frac{C_2}{(\alpha + i + k)^2}$$
$$+ \frac{D_2}{(\alpha - i - k)^2},$$

where

$$A = \frac{1}{2\alpha k^2} \cdot \frac{1}{(2\alpha - k)^2}, \quad C_2 = -\frac{1}{4\alpha^2 k} \cdot \frac{1}{2\alpha + k},$$

$$B = \frac{1}{2\alpha k^2} \cdot \frac{1}{(2\alpha + k)^2}, \quad D_2 = \frac{1}{4\alpha^2 k} \cdot \frac{1}{2\alpha - k}.$$

If $i = -k$, we have

$$\frac{A}{\alpha - k} + \frac{B}{\alpha + k} + \frac{C_1 + D_1}{\alpha} + \frac{C_2 + D_2}{\alpha^2} = \frac{1}{\alpha^4} \cdot \frac{1}{\alpha^2 - k^2}.$$

Thence

$$\alpha(A + B + C_1 + D_1) = \frac{\alpha k}{\alpha + k} B - \frac{\alpha k}{\alpha - k} A - (C_2 + D_2) + \frac{1}{\alpha^2(\alpha^2 - k^2)}$$
$$= -\frac{8\alpha^2 + k^2}{(4\alpha^2 - k^2)^2(\alpha^2 - k^2)} - \frac{1}{2\alpha^2(4\alpha^2 - k^2)} + \frac{1}{\alpha^2(\alpha^2 - k^2)}$$
$$= \frac{4}{(4\alpha^2 - k^2)^2} + \frac{2}{k^2(4\alpha^2 - k^2)} - \frac{1}{2\alpha^2 k^2}.$$

But, as before,

$$\Sigma \frac{1}{\alpha + i + \lambda} = \pi \cot \pi\alpha, \quad \Sigma \frac{1}{(\alpha + i + \lambda)^2} = \frac{\pi^2}{\sin^2 \pi\alpha}.$$

Hence

$$\Sigma_i\{i\}\{i+k\}^2 = \left[\frac{4}{(4\alpha^2-k^2)^2} + \frac{2}{k^2(4\alpha^2-k^2)} - \frac{1}{2\alpha^2k^2}\right]\frac{\pi\cot\pi\alpha}{\alpha} + \frac{\pi^2}{(4\alpha^2-k^2)2\alpha^2\sin^2\pi\alpha}$$

and

$$(4\alpha^2-1)\Sigma_i\{i\}\{i+k\}^2 = \left[\frac{4(k^2-1)}{(4\alpha^2-k^2)^2} + \frac{6k^2-2}{k^2(4\alpha^2-k^2)} + \frac{1}{2\alpha^2k^2}\right]\frac{\pi\cot\pi\alpha}{\alpha} + \left[1 + \frac{k^2-1}{(4\alpha^2-k^2)}\right]\frac{\pi^2}{2\alpha^2\sin^2\pi\alpha}.$$

Final value of $f(\alpha)$.—All the quantities in (5) can now be obtained by inserting the special values of k to find $d_1, \cdots, d_7$. Each term of the second line of (5) within the square brackets is to be multiplied by $4\alpha^2 - 1$ so that $(4\alpha^2-1)^{-3}$ appears as a factor of $f(\alpha)$. We also put

$$\frac{1}{\sin^2\pi\alpha} = \frac{2\cot\pi\alpha}{\sin 2\pi\alpha}, \quad \cot^2\pi\alpha = \frac{1+\cos 2\pi\alpha}{\sin 2\pi\alpha}\cot\pi\alpha,$$

so that $\cot\pi\alpha$ also appears as a factor of $f(\alpha)$. We obtain, finally,

$$f(\alpha) = \frac{\pi\cot\pi\alpha}{\alpha(4\alpha^2-1)^3}\left[-\frac{4\pi^2}{3\alpha^2} + \frac{\pi\cot 2\pi\alpha}{\alpha}\left(-\frac{2}{\alpha^2} - \frac{16}{4\alpha^2-1} - \frac{36}{4\alpha^2-4}\right)\right.$$
$$\left. + \frac{1}{\alpha^4} - \frac{25}{2}\cdot\frac{1}{\alpha^2} + \frac{64}{(4\alpha^2-1)^2} - \frac{32}{4\alpha^2-1} + \frac{144}{(4\alpha^2-4)^2} + \frac{18}{4\alpha^2-4} + \frac{64}{4\alpha^2-9}\right].$$

Hill's expression is derived from $f(\alpha)$ by replacing $4\alpha^2$ by Θ_0 and dividing by 2^{12}: this is the coefficient of $-\Theta_1^6$ in the expansion of his infinite determinant. When these changes are made, the above result agrees with that of Hill.

Some easy tests of its accuracy are available from the fact that $\alpha = \pm\frac{1}{2}$, $\alpha = \pm\frac{3}{2}$ cannot make $f(\alpha)$ infinite. If then we put $\alpha = \pm\frac{1}{2} + x$, $\alpha = \pm\frac{3}{2} + x$, and expand in powers of x, all negative powers must disappear. Further, since α does not occur in any term of the original expression for $f(\alpha)$ to a power higher than its square, it follows that $\lim \alpha^2 f(\alpha)$, $\alpha \doteq 0$, must be finite. Similarly $\lim (\alpha \pm 1)f(\alpha)$, $\alpha \doteq \mp 1$, must be finite. The expression has been found to satisfy all these tests.*

* Note added Jan. 24, 1912. It should be stated that J. C. Adams had also computed this series and had given the result in a different form (Mon. Not. Roy. Astr. Soc., vol. 48, p. 47), but with no indication of his method of procedure. An inspection of the extracts from his manuscripts (Coll. Works, vol. 2, p. 92), made since this paper was completed, shows that he followed methods similar to those adopted here.

New Haven, Conn.,
October 10, 1911.

A THEOREM IN DIFFERENCE EQUATIONS ON THE ALTERNATION OF NODES OF LINEARLY INDEPENDENT SOLUTIONS.

By E. J. Moulton.

Consider the difference equation

$$(1) \qquad L(i)u(i+1) + M(i)u(i) + N(i)u(i-1) = 0 \quad (i = 1, 2, \cdots, n).$$

Plot a solution $u(i)$ of this equation on an X-axis, making $u(i)$ correspond to a point x_i, where $x_i < x_{i+1}$, and joining successive points by straight line segments; a point x at which this broken line meets the axis will be called a *node* of the solution. Concerning the nodes of linearly independent solutions, the following analogon of a classic theorem of Sturm for differential equations may be stated:

THEOREM. *For any linear homogeneous difference equation of the form* (1), *where* $L(i)N(i) > 0$ *for every value of* i $(i = 1, 2, \cdots, n)$, *the nodes of any two linearly independent solutions separate each other.**

The proof of this theorem can be effected as follows. Let $u_1(i)$ and $u_2(i)$ be any two linearly independent solutions. Then

1°. No node of $u_1(i)$ coincides with a node of $u_2(i)$. For if x were a node of both $u_1(i)$ and $u_2(i)$ it would be a node of every solution; but supposing $x_k \leqq x \leqq x_{k+1}$, it is obvious that the solution determined by the conditions $u(k) = 1$, $u(k+1) = 1$ has no node at x.

Next, using the notation

$$(2) \qquad W(k) \equiv u_1(k+1)u_2(k) - u_1(k)u_2(k+1) \quad (k = 0, 1, \cdots, n),$$

it is known from the general theory of difference equations that $W(k)$ is distinct from zero; and one may prove

2°. For any values of k and l such that $W(k)$ and $W(k+l)$ are defined,

* The proof of this theorem was given for an exercise by Professor Bôcher in a course in differential equations at Harvard; and proofs more or less similar to the one here given were worked out simultaneously with and independently of this one by Messrs. Allen, Brand, Fort and Graustein. A theorem very similar to this one is given by Porter in these Annals, 2 Series, vol. 3, 1901–1902, on p. 65. He makes the assumption that neither of the solutions considered has a node at x_0. And his proof, which is very different in character from the one here given, depending upon the introduction of an auxiliary parameter, seems to be lacking in completeness; it does not seem obvious, for example, that the v-points (i. e., nodes) all vary in one direction as the parameter z increases.

$$\frac{W(k)}{W(k+l)} > 0.$$

It is no restriction to assume $l \geqq 0$; and for $l = 0$ the result is immediate. From the identities

$$L(k+1)u_1(k+2) + M(k+1)u_1(k+1) + N(k+1)u_1(k) = 0,$$

$$L(k+1)u_2(k+2) + M(k+1)u_2(k+1) + N(k+1)u_2(k) = 0,$$

one easily obtains

$$L(k+1)W(k+1) - N(k+1)W(k) = 0.$$

Hence, since $W(k+1) \neq 0$ and $L(k+1)N(k+1) > 0$,

$$\frac{W(k)}{W(k+1)} = \frac{L(k+1)}{N(k+1)} > 0;$$

and therefore

$$\frac{W(k)}{W(k+l)} = \prod_{j=1}^{l} \frac{L(k+j)}{N(k+j)} > 0.$$

The theorem is without content unless at least one of the solutions has two or more nodes on the interval $x_0 \leqq x \leqq x_{n+1}$. Suppose then that $u_1(i)$ has two successive nodes at x' and x'':

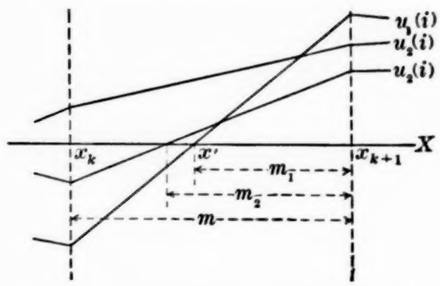
$$x_k \leqq x' < x_{k+1}, \quad x_{k+l} < x'' \leqq x_{k+l+1}, \quad l \geqq 1.$$

One may prove next

3°. If $u_2(i)$ has no node on the interval $x' < x < x''$ then

(3) $$a)\ \frac{W(k)}{u_1(k+1)u_2(k+1)} > 0; \quad b)\ \frac{W(k+l)}{u_1(k+l)u_2(k+l)} < 0.$$

Consider the interval $x_k \leqq x \leqq x_{k+1}$; the possibilities for $u_2(i)$ are indicated in the figure; it is to be observed that if the signs of $u_1(i)$ or of $u_2(i)$



or of both $u_1(i)$ and $u_2(i)$ are changed the following argument is not altered. Now $u_1(k+1)$ and $u_2(k+1)$ are not zero and hence from (2)

$$\frac{W(k)}{u_1(k+1)u_2(k+1)} = \frac{u_2(k)}{u_2(k+1)} - \frac{u_1(k)}{u_1(k+1)}.$$

If $u_1(k) = 0$, then, using 1°,

$$\frac{u_2(k)}{u_2(k+1)} > 0;$$

hence (3*a*). If $u_1(k) \neq 0$, then either

$$\frac{u_2(k)}{u_2(k+1)} \geqq 0, \quad \frac{-u_1(k)}{u_1(k+1)} > 0,$$

and hence (3*a*); or, referring to the figure, and using 1°,

$$\frac{u_2(k)}{u_2(k+1)} - \frac{u_1(k)}{u_1(k+1)} = \left(-\frac{m-m_2}{m_2}\right) - \left(-\frac{m-m_1}{m_1}\right) = -m\frac{m_1-m_2}{m_1m_2} > 0.$$

Therefore (3*a*) holds in all cases. (3*b*) can be established in a similar manner.

Now suppose that $u_2(i)$ has no node on the interval $x' < x < x''$; then dividing (3*a*) by (3*b*),

$$\frac{W(k)}{W(k+l)} \cdot \frac{u_1(k+l)}{u_1(k+1)} \cdot \frac{u_2(k+l)}{u_2(k+1)} < 0.$$

Hence by 2° and the hypothesis that $u_1(i)$ has no node on the interval $x' < x < x''$, it follows that

$$\frac{u_2(k+l)}{u_2(k+1)} < 0.$$

If $l = 1$ this leads at once to a contradiction; if $l > 1$ this implies that $u_2(i)$ has a node on $x_{k+1} \leqq x \leqq x_{k+l}$, and one is again led to a contradiction. Hence between every pair of nodes of one solution there is a node of the other; that is, the nodes separate each other.

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PERIODIC QUADRATIC TRANSFORMATIONS IN THE PLANE.

By Virgil Snyder.

In his prize essay on periodic transformations, Kantor * mentions one class in which each power of a quadratic transformation is quadratic, unless it is periodic, in which case each power except the identity is a quadratic transformation.† No equations are given, and but few of the properties of the transformation are derived. It is probably for this reason that an important category of quadratic transformations is omitted from Kantor's list, and consequently whole groups "with two points" should be added to Wiman's classification.‡ There is no error in the latter paper, but the results of Kantor's investigations are assumed to be correct. By applying the same methods, as used by Wiman, upon the omitted forms a complete list can be obtained. It is the purpose of this paper to obtain the equation, define the system of fundamental elements, and discuss some of the properties of these transformations, and to show the existence of new ones not heretofore considered.‖

1. Let $x_i' = \varphi_i(x_1, x_2, x_3) = \varphi_i(x)$ $[i = 1, 2, 3]$ define a birational quadratic transformation T of period 3 in a ternary field. Let A, B, C be the fundamental points in $[x]$, and A', B', C' those in $[x']$ such that the image of A is $B'C'$, etc. Finally, suppose that x_i, x_i' are referred to the same triangle of reference and the same unit point. An arbitrary line $c_1(x)$ will go into a conic $c_2(x')$ through A', B', C'. This conic, regarded as in $[x]$, will have for image in $[x']$ a rational quartic having double points at A', B', C'. On the other hand, from the relation $T^3 = 1$ or $T^2 = T^{-1}$ it follows that the line c_1, regarded as in $[x']$, will go into a conic circumscribing the triangle ABC. This conic must be identical with the quartic with double points at A', B', C', after extraneous factors have been removed. This condition

* S. Kantor: Premiers fondements pour une théorie des transformations periodiques univoques; mémoire couronnée par l'académie des sciences physiques et mathématiques de Naples dans le concours pour 1883, Atti della R. accademia delle scienze di Napoli, ser. 2, vols. 3 and 4 (1891), pp. 1–356.

† L. c., § 30.

‡ A. Wiman: Ueber die endlichen Gruppen von birationalen Transformationen in der Ebene, Mathematische Annalen, vol. 48 (1895), pp. 195–235.

‖ For the definitions of the various words used throughout the paper as well as for a systematic treatment of the elementary properties of these transformations, see Miss Scott's Introduction to modern analytic geometry (1892), pp. 218–225, or Doehlemann, Geometrische Transformationen, Zweiter Teil (1908), pp. 1–23.

necessitates that *the two triangles ABC, $A'B'C'$ have two and only two vertices in common,* and that the third vertex of each can not lie on any side of the other, since the passage of a curve through a fundamental point is the necessary and sufficient condition that its image should contain a fundamental straight line as a factor.

We shall first assume that the points A, B, C are distinct. The cases in which two or all three approach coincidence will be considered later. Several cases may now arise, but all the others may be derived from one by multiplying a certain quadratic transformation by an appropriate cyclic collineation. Assume $B \equiv B'$, $C \equiv C'$.

2. The image of the point A is, by hypothesis, the line $B'C' = BC$. The image of BC is, by hypothesis, the point A'. Since $T^3 = 1$, it follows that the image of A', regarded as a point of $[x]$, is A, regarded as a point of $[x']$.

Since no three of these points are collinear and no two approach coincidence, we may take, without loss of generality

$$A \equiv (1, 0, 0), \quad B \equiv (0, 1, 0), \quad C \equiv (0, 0, 1), \quad A' \equiv (1, 1, 1).$$

Since by T a straight line in $[x]$ goes into a conic in $[x']$ through A', $B' \equiv B$, $C' \equiv C$, we may write

$$\begin{aligned}
\rho x_1 &= a(x_1'^2 - x_2'x_3') + bx_2'(x_1' - x_3') + cx_3'(x_1' - x_2'),\\
\rho x_2 &= a'(x_1'^2 - x_2'x_3') + b'x_2'(x_1' - x_3') + c'x_3'(x_1' - x_2'),\\
\rho x_3 &= a''(x_1'^2 - x_2'x_3') + b''x_2'(x_1' - x_3') + c''x_3'(x_1' - x_2').
\end{aligned}$$

Any line $m_3x_2 - m_2x_3 = 0$ through A goes into a conic having $x_1' = 0$ as factor, for every value of $m_2 : m_3$, hence

$$\begin{aligned}
a' + b' + c' &= 0,\\
a'' + b'' + c'' &= 0.
\end{aligned}$$

In the same way, since a line $m_3x_1 - m_1x_3 = 0$ through B goes into a conic having $x_1' - x_2' = 0$ as factor, we obtain the further relations

$$a + b = 0, \quad a'' + b'' = 0.$$

A line $m_2x_1 - m_1x_2 = 0$ goes into a conic having $x_1' - x_3' = 0$ as factor, from which

$$a + c = 0, \quad a' + c' = 0.$$

The point $(1, 0, 0)$ in $[x']$ has $(1, 1, 1)$ for its image in $[x]$, hence

$$a = a' = a''.$$

These eight equations of condition are just sufficient to determine the coefficients in the equations of the transformation. The result is:

$$T:\quad \begin{aligned} \sigma x_1' &= x_2x_3,\\ \sigma x_2' &= x_3(x_2 - x_1),\\ \sigma x_3' &= x_2(x_3 - x_1),\end{aligned}\qquad T^{-1}:\quad \begin{aligned} \rho x_1 &= (x_1' - x_2')(x_1' - x_3'),\\ \rho x_2 &= x_1'(x_1' - x_3'),\\ \rho x_3 &= x_1'(x_1' - x_2').\end{aligned}$$

3. The points of the plane are arranged in triads. If the points of a triad are collinear, the locus containing the triads will be invariant under the transformation. The equation of the locus is

$$\begin{vmatrix} x_1 & x_2 & x_3\\ x_2x_3 & x_3(x_2 - x_1) & x_2(x_3 - x_1)\\ (x_1 - x_2)(x_1 - x_3) & x_1(x_1 - x_3) & x_1(x_1 - x_2)\end{vmatrix} = 0;$$

it consists of the five lines

$$x_2-x_3=0,\quad x_1-\theta x_3=0,\quad x_1+\theta^2x_3=0,\quad x_1-\theta x_2=0,\quad x_1+\theta^2x_2=0\quad [\theta^3=-1].$$

Of these, the first, second, and fourth meet at $D_1 \equiv (\theta, 1, 1)$, while the first, third, and fifth meet at $D_2 \equiv (-\theta^2, 1, 1)$. Thus the invariant line $x_2 - x_3 = 0$ contains two of the self-corresponding points. The others are $D_3 \equiv (\theta, \theta^2, -1)$, the intersection of $x_1 - \theta x_3 = 0$, $x_1 + \theta^2 x_2 = 0$ and $D_4 \equiv (\theta, 1, \theta^2)$, the intersection of $x_1 - \theta x_2 = 0$, $x_1 + \theta^2 x_3 = 0$.

4. The pencil of lines through C remains invariant, a triad being formed by the lines

$$x_2 = kx_1,\quad x_2 = \frac{x_1}{1-k},\quad x_2 = \frac{k-1}{k}x_1.$$

Similarly for the pencil through B.

5. In the preceding equations of condition among the coefficients, the first six were obtained independently of the periodicity. If $T^n = 1$, the last two equations are replaced by $A'T^{n-2}A$. All the coefficients are linear functions of a, a', a''; of these, a', a'' enter symmetrically. The equations now become

$$T:\quad \begin{aligned} \sigma x_1' &= ax_2x_3,\\ \sigma x_2' &= x_3(ax_2 - a'x_1),\\ \sigma x_3' &= x_2(ax_3 - a''x_1),\end{aligned}\qquad T^{-1}:\quad \begin{aligned} \rho x_1 &= a(x_1' - x_2')(x_1' - x_3'),\\ \rho x_2 &= a'x_1'(x_1' - x_3'),\\ \rho x_3 &= a''x_1'(x_1' - x_2').\end{aligned}$$

If we put $a' = a'' = 1$, the following properties are now apparent. The pencils of straight lines through B and through C are invariant. The transformation is a collineation for each of these pencils, leaving two

lines invariant. Those of B are $x_1 - m_1x_3 = 0$, $x_1 - m_2x_3 = 0$, m_1, m_2 being defined by

$$m^2 - am + a = 0.$$

The lines $x_1 - m_1x_2 = 0$, $x_1 - m_2x_2 = 0$ are invariant through C. The line $AA' \equiv x_2 - x_3 = 0$ is invariant and contains all the $n - 2$ images of A'.

The lines $x_1 - m_1x_2 = 0$, $x_1 - m_1x_3 = 0$ intersect in D_1, $x_1 - m_2x_2 = 0$, $x_1 - m_2x_3 = 0$ intersect in D_2; D_1, D_2 both lie on $x_2 - x_3 = 0$. The lines $x_1 - m_1x_3 = 0$, $x_1 - m_2x_2 = 0$ intersect in D_3; $x_1 - m_2x_3 = 0$, $x_1 - m_1x_2 = 0$ intersect in D_4. The points

$$D_1 \equiv (m_1m_2, m_2, m_2), \quad D_2 \equiv (m_1m_2, m_1, m_1),$$

$$D_3 \equiv (m_1m_2, m_1, m_2), \quad D_4 \equiv (m_1m_2, m_2, m_1)$$

are the only invariant points of the transformation T. Every conic $k_1(x_1 - m_1x_3)(x_1 - m_2x_2) - k_2(x_1 - m_1x_2)(x_1 - m_2x_3) = 0$ of the pencil through B, C, D_1, D_2 remains invariant. Thus, *any point P and all its images under T lie on the same conic of the pencil, when $a' = a''$.*

6. The value of a can be readily determined when n is given. If $x_1^{(k)}$, $x_2^{(k)}$, $x_3^{(k)}$ are the coordinates of the kth image of (x_1, x_2, x_3) when operated upon by T, then the point $A' \equiv (1, 1, 1)$ becomes

$$x_1^{(k)} = \beta'\cdot\beta'', \quad x_1^{(k+1)} = \gamma'\cdot\gamma'',$$

$$x_2^{(k)} = \beta''\cdot\gamma', \quad x_2^{(k+1)} = \gamma''\cdot\delta',$$

$$x_3^{(k)} = \beta'\cdot\gamma'', \quad x_3^{(k+1)} = \gamma'\cdot\delta'',$$

wherein $\delta' = \gamma' - a'\beta'$, $\delta'' = \gamma'' - a''\beta''$, each function l'' being the same function of a and a'', as l' is of a and a'. If

$$k = 2,$$

then

$$\beta' = a - a',$$

$$\gamma' = a - 2a';$$

putting $a' = a'' = 1$, the following formulas are obtained:

$$\begin{aligned} a &= 1 & &\text{for } n = 3 \\ a &= 2 & &n = 4 \\ a^2 - 3a + 1 &= 0 & &n = 5 \\ a^2 - 4a + 3 &= 0 & &n = 6 \\ a^3 - 5a^2 + 6a - 1 &= 0 & &n = 7 \\ a^3 - 6a^2 + 10a - 4 &= 0 & &n = 8 \\ \cdots\cdots & & &\cdots \end{aligned}$$

For any composite value of n, one root a of the associated equation belongs also to the equation associated with that factor. All the roots of the equation are real.

7. The fundamental triangles all have the vertices B, C in common; the third vertex of each always lies on the invariant line $x_2 - x_3 = 0$.

$$\begin{array}{lll} \text{For } n = 3, & T \text{ has } ABC \text{ in } [x], & A'BC \text{ in } [x'], \\ & T^2 \text{ has } A'BC \text{ in } [x], & ABC \text{ in } [x']. \\ \text{For } n = 4, & T \text{ has } ABC \text{ in } [x], & A'BC \text{ in } [x'], \\ & T^2 \text{ has } A_1'BC \text{ in } [x], & A_1'BC \text{ in } [x'], \\ & T^3 \text{ has } A'BC \text{ in } [x], & ABC \text{ in } [x'], \end{array}$$

wherein $A'TA_1'$, $A_1' \equiv (2, 1, 1)$.

The images of A' are thus arranged in pairs, $A'T^kA_k'$, $A'T^{n-k}A_{n-k}'$ being associated. When $n = 2m$, in T^m the fundamental triangles must coincide, since $(T^m)^2 = 1$, or $T^m = T^{-m}$.

8. For $n = 4$, $a = 2$, $m_1 = 1 + i$, $m_2 = 1 - i$. There are two pencils of cubics which remain invariant. The first is

$$k_1(x_1 - m_1x_3)(x_1 - m_2x_2)(x_2 - x_3) + k_2[(1 - m_2)x_1(x_2 - x_3) + (m_1 - m_2)\,x_3(x_1 - x_2)]\cdot l = 0,$$

wherein

$$l = m_1m_2(x_2 - x_3) + 2(m_1x_3 - m_2x_2) + x_1(m_2 - m_1) = 0.$$

The other is obtained from it by interchanging m_1, m_2. The first is composed of cubics having a node at D_3. The second is composed of harmonic curves.

9. When n is a prime greater than 3, the associated value of a is irrational.

10. The equations of the transformation will now be obtained from a different point of view, without any use of the preceding method. Since the pencils of straight lines through B and C are invariant, the transformation can be generated most easily by the Seydewitz method.*

The line $x_3 = 0$ goes over into $x_1' = 0$, $x_1 = 0$ into $x_1' - x_3' = 0$. The line $x_1y_3 - x_3y_1 = 0$ through the point (y) goes into

$$x_1'(ky_3 - y_1) = ky_3x_3',$$

k being an undefined parameter.

* Darstellung der geometrischen Verwandtschaften . . . , Archiv der Mathematik und Physik, vol. 7 (1846), pp. 113–148.

The values of m_1, m_2 in the equations of the invariant lines of the pencil

$$x_1 - m_1x_3 = 0, \quad x_1 - m_2x_3 = 0$$

are roots of the equation

$$m^2 - km + k = 0.$$

The characteristic anharmonic ratio of this linear transformation is m_1/m_2. If the period is n, then

$$m_1^n = m_2^n.$$

Since

$$m_1 + m_2 = k, \quad m_1m_2 = k, \quad m_1 \neq m_2,$$

we have an equation of order $E[(n-1)/2]$ to determine k, $E(s)$ being the largest integer not greater than s. It is exactly the preceding equation in a.

The same equations will hold for C if x_3 is replaced by x_2 and k by l. The period of this pencil may be n or any other positive integer. The line $x_1y_2 - x_2y_1 = 0$ joining (y) to C goes into $x_1(ly_2 - y_1) = y_2x_2$, hence (y'), the image of (y), is defined by the intersection of the lines

$$x_1(ly_2 - y_1) = ly_2x_2, \quad x_1(ky_3 - y_1) = ky_3x_3.$$

Replacing (y), (y') by (x), (x'), the equations of the transformation become

$$S:\ \begin{aligned} \sigma x_1' &= klx_2x_3, \\ \sigma x_2' &= kx_2(lx_3 - x_1), \\ \sigma x_3' &= lx_3(kx_2 - x_1), \end{aligned} \qquad S^{-1}:\ \begin{aligned} \rho x_1 &= kl(x_1' - x_2')(x_1' - x_3'), \\ \rho x_2 &= lx_1'(x_1' - x_2'), \\ \rho x_3 &= kx_1'(x_1' - x_3'). \end{aligned}$$

If $kl = a$, $l = a'$, $k = a''$, S is the product of T and the harmonic homology H: $x_1 = \sigma x_1'$, $x_2 = \sigma x_3'$, $x_3 = \sigma x_2'$, operating first with T and then with H. $S = TH$.

11. If $k = l$, the new equations of T reduce to the form previously derived. If $k \neq l$, the successive images of A' are not all on the line $x_2 - x_3 = 0$, nor do the images of a point P all lie on a conic of the pencil BCD_1D_2. The pencil $B(PP'D_1D_2)$ has the anharmonic ratio $m_1 : m_2$, while the pencil $C(PP'D_1D_2)$ has the ratio $m_1' : m_2'$. Even if the linear transformations of the pencils B and C have the same period $n(n > 3)$, k, l may be taken as different roots of the same equation. When n is prime, there are therefore $[(n-1)/2]^2$ different transformations of period n, but of these only $[(n-1)/2]$ have the images of a point all lie on a conic.

12. In Kantor's treatment, which is purely synthetic, the case $k \neq l$ is not considered. In consequence, all the finite groups generated by T and any other finite groups are omitted from his list, and also from that of Wiman, although the methods used by the latter are correct.

13. Transformations of the form T, $(k \neq l)$, cannot be regarded as the

plane depiction of a linear transformation in space under which a quadric surface remains invariant.

14. The remaining cases can all be obtained from T by first performing a cyclic linear transformation.

If $B' \equiv C$, $C' \equiv B$, first apply the harmonic homology $\sigma x_1 = a'a''x_1'$, $\sigma x_2 = a'^2x_3'$, $\sigma x_3 = a''^2x_2'$, then T.

If $A' \equiv B$, $C' \equiv C$, apply

$$\sigma x_1 = a^2x_2', \quad \sigma x_2 = a'^2x_1, \quad \sigma x_3 = aa'x_3'.$$

If $A' \equiv C$, $B' \equiv B$, apply

$$\sigma x_1 = a^2x_3', \quad \sigma x_2 = aa''x_2', \quad \sigma x_3 = a''^2x_3'.$$

If $A' \equiv B$, $B' \equiv C$, apply

$$\sigma x_1 = a^2a'x_3', \quad \sigma x_2 = a'^2a''x_1', \quad \sigma x_3 = aa''^2x_2'.$$

If $A' \equiv C$, $C' \equiv B$, apply

$$\sigma x_1 = a^2a''x_2', \quad \sigma x_2 = aa'^2x_3', \quad \sigma x_3 = a'a''^2x_1'.$$

The last two linear transformations are cyclic of order 3.

15. When two of the fundamental points approach coincidence, the same thing happens to the inverse, and we have a quadratic transformation of the second kind.* Let the images of the straight lines of the plane be a net of conics touching a line p at the point P, and also passing through another point Q. In order that each power of this transformation be quadratic, the inverse system of conics must have two of these basis points in common, that is, they must either pass through P and Q or touch p at P. In the former case they can have a common tangent p' at P ($p' \neq p$), or all touch a common line q at Q. In the second case they also have another basis point R. These three cases will be considered in turn.

16. Let the inverse system pass through P and touch q at Q. Put

$$p \equiv x_3 = 0, \quad q \equiv x_2 = 0, \quad PQ \equiv x_1 = 0.$$

A simple calculation shows that the only possible form is

$$\begin{aligned} \sigma x_1' &= ax_1x_3, & \rho x_1 &= acx_1'x_2', \\ \sigma x_2' &= bx_1^2, & \rho x_2 &= a^2x_2'x_3', \\ \sigma x_3' &= cx_2x_3, & \rho x_3 &= bcx_1'^2. \end{aligned}$$

It is of period four, independent of a, b, c; its square is the involutorial quadratic transformation of the first kind.

* Doehlemann, l. c., p. 30; Scott, l. c., p. 222.

17. Let the inverse system touch p' at P and pass through Q. The equations become

$$\begin{aligned} \sigma x_1' &= a x_1 x_3, & \rho x_1 &= c x_1'(a x_3' - k x_1'), \\ \sigma x_2' &= c x_2 x_3, & \rho x_2 &= a x_2'(a x_3' - k x_1'), \\ \sigma x_3' &= d x_1^2 + k x_1 x_3, & \rho x_3 &= c d x_1'^2. \end{aligned}$$

The two projectivities are defined by

$$m_1' = \frac{a}{c} m_1, \quad m_2' = \frac{d + k m_2}{a m_2},$$

wherein

$$m_1 x_2 = x_1, \quad m_2 x_1 = x_3.$$

In this case transformations of any period greater than 2 may appear. The common tangent is $p' \equiv a x_3' - k x_1'$.

18. Let the inverse system touch p at P and pass through R. The general form of the transformation is

$$\begin{aligned} \sigma x_1' &= a x_1 x_3 + b x_1^2, \\ \sigma x_2' &= c x_2 x_3, \\ \sigma x_3' &= d x_1^2. \end{aligned}$$

the new basis point being $R \equiv (b, 0, d)$.

From the equations we have the two projectivities

$$\frac{d x_1' - b x_3'}{a x_3'} = \frac{x_3}{x_1}, \tag{1}$$

$$\frac{a x_2'}{d x_1' - b x_3'} = \frac{c x_2}{d x_1}. \tag{2}$$

In the first, the two pencils are concentric, the invariant lines are $x_3 = m x_1$, wherein m is defined by the quadratic equation

$$a m^2 + b m - d = 0.$$

The second is a perspectivity between the lines of two non-concentric pencils, the axis passing through $P \equiv (0, 1, 0)$. By means of the figure suggested by (1) and (2) it is easily seen that *this transformation cannot be periodic for any values of a, b, c, d, except when $b = 0$*, in which case it does not belong to this type.

19. In case all three basis points approach coincidence the general form becomes *

* Doehlemann, l. c., p. 32; Scott, l. c., p. 222.

$$\rho x_1' = a(x_1x_2 - mx_3^2),$$

$$\rho x_2' = bx_2^2,$$

$$\rho x_3' = cx_2x_3.$$

The inverse is already in the proper form. By repeating the transformation the law of composition of the coefficients is at once seen. The transformation will be periodic, of period n, if

$$b = 1, \quad a^n = c^n = (ac)^n = 1, \quad a \neq 1, \quad c \neq 1.$$

There are as many types as there are sets of values of a and c satisfying these conditions. For $n > 5$ a point and its successive images will lie on a conic when and only when $a = c$. Neither Kantor nor Wiman considers any of the cases in which fundamental points approach coincidence.

CORNELL UNIVERSITY,
August, 1911.

ON THE REDUCTION OF A SYSTEM OF LINEAR DIFFERENTIAL FORMS OF ANY ORDER.

BY ARNOLD DRESDEN.

In the classical theory of the extremum of a definite integral of the form $\int F(x, y, x', y')dt$, the discussion of the second variation and of Jacobi's differential equation is based on the reduction of the second variation by means of Weierstrass's transformation to a form analogous to that of the second variation of the integral $\int f(x, y, y')dx$.* It appears to the writer that the essential character of Weierstrass's transformation is the reduction of linear differential forms of the second order in three variables to similar forms in two variables. This character becomes more apparent when one attempts to parallel Weierstrass's work for the case of an integrand containing n unknown functions of t and their first derivatives.

The purpose of the present note is to consider, from a general point of view, the problem of reducing a system of n linearly independent differential forms of pth order, containing $n + 1$ unknown functions of t, to a similar system containing n unknown functions. The application to the calculus of variations can then be made by putting $p = 2$. So far as I have been able to determine, this problem has not previously been discussed in the literature.

Suppose we have n differential forms:

$$(1)\qquad \psi_i \equiv \sum_{j=1}^{n+1} \sum_{l=0}^{p} a_{ilj} x_j^{(l)} \qquad (i = 1 \cdots n),$$

where a_{ilj} are functions of t of class $C^{(l)}$† on an interval $(t_1 t_2)$ and such that the matrix $\| a_{ipj} \|$ is of rank n.‡

We shall then seek to determine α_{kj} and A_{ilk} in such a way that we can write

$$(2)\qquad \psi_i \equiv \sum_{k=1}^{n} \sum_{l=0}^{p} A_{ilk} y_k^{(l)},$$

where

$$(3)\qquad y_k \equiv \sum_{j=1}^{n+1} \alpha_{kj} x_j.$$

* Compare, e. g., Bolza, Vorlesungen über Variationsrechnung, §§ 28 and 29.

† A function is said to be of class $C^{(l)}$ if the function is continuous and has continuous derivatives of the 1st, 2d, . . ., lth order.

‡ For the definition of rank, see Bôcher, Introduction to Higher Algebra, p. 22.

If we suppose $|A_{ipk}| \neq 0$, and put

$$(4) \qquad z_i = \sum_{k=1}^{n} A_{ipk} y_k,$$

we can solve the latter equations for y_k in terms of z_i; substituting the results in (2) would reduce these expressions to the form

$$(5) \qquad \psi_i \equiv z_i^{(p)} + \sum_{k=1}^{n} \sum_{h=0}^{p-1} B_{ihk} z_k^{(h)};$$

and, substitution of (4) in (5) would reduce these back to (2).

Hence there is no loss in generality if we try to determine β_{kj} and B_{ihk} in such a way as to reduce (1) to the form (5) directly, where

$$(6) \qquad z_k \equiv \sum_{j=1}^{n+1} \beta_{kj} x_j.$$

Substituting (6) in (5) and comparing the coefficients of the terms in $x_j^{(p)}$, with the corresponding coefficients in (1), we see at once that

$$\beta_{kj} \equiv a_{kpj}$$

for all values of k and j. Thence we have

$$(7) \qquad z_i = \sum_{j=1}^{n+1} a_{ipj} x_j.$$

Rather than compare coefficients of terms of lower order, we proceed now as follows: Since the matrix $\| a_{ipj} \|$ was supposed to be of rank n, equations (7) can be solved for x_j in terms of z_i; we find:

$$x_j = \sum_{k=1}^{n} C_{ji} z_i + \Delta_j w,$$

where Δ_j is the n-rowed determinant obtained from $\| a_{ipj} \|$ by striking out the jth column and multiplying the result by $(-1)^{j-1}$, w being an arbitrary function of t.

If we substitute these results in (1), these forms will reduce to forms (5), *if, and only if, the terms in w vanish identically.*

This leads at once to the following necessary and sufficient conditions for the possibility of our reduction:

$$(8) \qquad \sum_{j=0}^{n+1} \sum_{l=0}^{p} a_{ilj} (\Delta_j w)^{(l)} \underset{w}{\equiv} 0, \qquad (i = 1, \cdots, n).$$

In order to express these conditions in terms of the coefficients of (1) alone, we make use of the following formula:

$$uv^{(l)} \equiv \sum_{m=0}^{l} (-1)^m C_{l,\,m} (u^{(m)} v)^{(l-m)}, \tag{9}$$

where

$$C_{l,\,m} \equiv \frac{l!}{m!\,(l-m)!},$$

which we can prove by a complete induction.

We know $uv^{(l)} \equiv (uv^{(l-1)})' - u'v^{(l-1)}$.

Assuming the formula to hold for $l - 1$, we have:

$$uv^{(l-1)} \equiv \sum_{m=0}^{l-1} (-1)^m C_{l-1,\,m} (u^{(m)} v)^{(l-m-1)};$$

$$\begin{aligned} u'v^{(l-1)} &\equiv \sum_{m=0}^{l-1} (-1)^m C_{l-1,\,m} (u^{(m+1)} v)^{(l-m-1)} \\ &\equiv \sum_{m=1}^{l} (-1)^{m-1} C_{l-1,\,m-1} (u^{(m)} v)^{(l-m)}. \end{aligned}$$

Hence

$$\begin{aligned} uv^{(l)} &\equiv \sum_{m=0}^{l-1} (-1)^m C_{l-1,\,m} (u^{(m)} v)^{(l-m)} + \sum_{m=1}^{l} (-1)^m C_{l-1,m-1} (u^{(m)} v)^{(l-m)} \\ &\equiv \sum_{m=0}^{l} (-1)^m [C_{l-1,\,m} + C_{l-1,\,m-1}] (u^{(m)} v)^{(l-m)} \\ &\equiv \sum_{m=0}^{l} (-1)^m C_{l,\,m} (u^{(m)} v)^{(l-m)}, \end{aligned}$$

which proves formula (9) for l. Since we can verify by immediate substitution that the formula holds for $l = 2, 3$, its validity has been established.

By means of (9), conditions (8) reduce to:

$$\sum_{j=1}^{n+1} \sum_{l=0}^{p} \sum_{m=0}^{l} (-1)^m C_{l,\,m} [a_{ilj}{}^{(m)} \Delta_j w]^{(l-m)} \equiv 0, \qquad (i = 1 \cdots n).$$

Putting $l - m = r$, these become

$$\sum_{j=1}^{n+1} \sum_{r=0}^{p} \sum_{l=r}^{p} (-1)^{l-r} C_{l,\,l-r} [a_{ilj}{}^{(l-r)} \Delta_j w]^{(r)} \equiv 0, \qquad (i = 1 \cdots n).$$

Again, since these conditions have to hold true for any function w, we obtain the following $pn + n$ conditions,

$$\sum_{j=1}^{n+1} \sum_{l=r}^{p} (-1)^{l-r} C_{l,\,l-r} a_{ilj}{}^{(l-r)} \Delta_j \equiv 0 \qquad (r = 0, \cdots, p;\ i = 1, \cdots, n). \tag{10}$$

For $r = p$, these reduce to

$$\sum_{j=1}^{n+1} a_{ipj} \Delta_j \equiv 0 \qquad (i = 1, \cdots, n),$$

which are satisfied in virtue of the definition of Δ_j. The remaining pn conditions can be expressed by equating to zero certain $(n+1)$-rowed determinants. The first n rows of these are always formed by the matrix $\| a_{ipj} \|$; the last row is formed by the elements

$$\sum_{l=r}^{p} (-1)^{l-r} C_{l,\,l-r} a_{ilj}{}^{(l-r)} \qquad (j = 1, \cdots, n+1).$$

For the particular case of 2 forms of the second order involving 3 unknown functions of t, the conditions take the form

$$\begin{vmatrix} a_{131} & a_{132} & a_{133} \\ a_{231} & a_{232} & a_{233} \\ 2a_{i31}' - a_{i21} & 2a_{i32}' - a_{i22} & 2a_{i33}' - a_{i23} \end{vmatrix} = 0 \qquad (i = 1, 2),$$

$$\begin{vmatrix} a_{131} & a_{132} & a_{133} \\ a_{231} & a_{232} & a_{233} \\ a_{i31}'' - a_{i21}' + a_{i11} & a_{i32}'' - a_{i22}' + a_{i12} & a_{i33}'' - a_{i23}' + a_{i13} \end{vmatrix} = 0 \qquad (i = 1, 2).$$

We can now state the following conclusion:

If we have given a system of differential forms (1), *in which the coefficients a_{ilj} are functions of t of class $C^{(l)}$ on (t_1t_2) such that the matrix $\| a_{ipj} \|$ is of rank n, and if these coefficients satisfy, on that interval, the pn conditions* (10) $(r = 0, 1, \cdots, p-1;\ i = 1, \cdots, n)$, *then n^2 functions of t, A_{ipk}, may be chosen arbitrarily except for the condition $| A_{ipk} | \neq 0$ on (t_1t_2); and $(p+1)n^2 + n$ functions of t, A_{ilk} and α_{kj}, are then determined uniquely, in such a way that the forms* (1) *are reducible to the system* (2).

In the application of the reduction discussed in this note, it is frequently desirable, after its possibility has been established, to be able to select arbitrarily the coefficients α_{kj} in equations (3).

To determine the conditions under which this is possible, we substitute (3) in (2) and compare the coefficients of $x_j{}^{(p)}$ with the corresponding coefficients in (1). This leads to i systems of equations, each system consisting of $n+1$ equations in the n unknowns A_{ipk}:

$$\sum_{k=1}^{n} A_{ipk}\alpha_{kj} = a_{ipj} \qquad \begin{matrix} (i = 1, \cdots, n), \\ (j = 1, \cdots, n+1). \end{matrix}$$

These will be solvable for A_{ipk} if the i eliminants vanish. It is furthermore clear, that if α_{kj} are finite, the values of A_{ipk} so determined satisfy the condition $| A_{ipk} | \neq 0$, required in our theorem.

Thence, if conditions (10) are satisfied, the coefficients α_{kj} may be selected arbitrarily and all the coefficients of the new forms determined uniquely, but for a constant factor, if and only if we also have:

$$\sum_{j=1}^{n=1} a_{ip_j} D_j = 0 \qquad (i = 1, \cdots, n), \tag{11}$$

where D_j is the determinant obtained from the matrix $|\alpha_{kj}|$ by striking out the jth column and multiplying by $(-1)^j$.

UNIVERSITY OF WISCONSIN,
December, 1911.

ON THE FUNCTIONAL EQUATION FOR THE SINE. ADDITIONAL NOTE.

By Edward B. Van Vleck.

In vol. 10 of the Annals, p. 161, I brought forward the following functional equation for the sine:

$$\text{(I)} \qquad f(x - y + A) - f(x + y + A) = 2f(x)f(y).$$

In the second of the two proofs there given for the definition of the sine by means of this equation, one step should have been supplied to the reader between 7) and 8) to justify in the second equation of 8) the exclusion of the values $f(x) = \pm 1$ for $0 < x < B < A$. The justification of this exclusion runs as follows.

Suppose k to be a value between 0 and A for which $f(k) = \pm 1$. Then by my equation * (2),

$$f(k + A) = 0.$$

Putting $y = k$ in (1) we have

$$f(x - k + A) + f(x + k + A) = 0.$$

But for $y = k$ equation (I) becomes

$$f(x - k + A) - f(x + k + A) = \pm 2f(x).$$

The combination of these two equations gives

$$f(x + k + A) = \mp f(x).$$

This is equivalent to

$$f(x + k - A) = \pm f(x),$$

since the addition of $2A$ to the argument of the function changes its sign. Hence if x is replaced in (I) by $x + k - A$, there results the equation

$$f(x - y + k) - f(x + y + k) = \pm 2f(x)f(y),$$

where the upper or lower sign holds in the right hand member according as $f(x) = + 1$ or $f(x) = - 1$. In the former case the equation is the same as (I) with k in place of A, while in the latter case we have merely to put $f_1(x) \equiv - f(x)$ to make it the same. We may therefore suppose A in (I) to be the smallest value of $k > 0$ for which $f(k) = \pm 1$. In other words, between 0 and A we have

$$- 1 < f(x) < 1.$$

* The equations referred to are

$$(1) \qquad f(x - y + A) + f(x + y + A) = 2f(x + A)f(y + A),$$

$$(2) \qquad 1 = f^2(x) + f^2(x + A).$$

ON THE RECTILINEAR CONGRUENCE REALIZING A CIRCULAR TRANSFORMATION OF ONE PLANE INTO ANOTHER.*

By Arnold Emch.

1. In what follows I shall deal with the rectilinear (2, 2) congruence constructed by means of a circular correspondence between two planes. As far as I am aware the construction of the congruence has not previously been specialized in this way.

The general construction of congruences by means of birational correspondences, and the converse, has been studied by several authors. Steiner,† for example, established a quadratic correspondence between two planes by the intersection of the congruence of lines through any two skew-lines with those planes. Hirst,‡ on the other hand, investigated the class of congruences, based upon the Cremona transformations of two planes.

2. Consider a sphere of radius r and center M and on it any two points Z and Z' with the distance $ZZ' < 2r$ and any two planes $E \perp ZM$ and $E' \perp Z'M$, Fig. 1. E and E' may be considered as inversions of the sphere from Z and Z' as centers. Take any point P on the sphere and let A and A' be the inverses of P in the planes E and E'. Let B and B' designate the piercing-points of the straight line through Z and Z' with E and E' and let $ZZ' = s$, $ZB = p$, $Z'B' = p'$, $BB' = q$, so that $q = p + p' - s$; then

$$
\begin{aligned}
ZP \cdot ZA &= ZZ' \cdot ZB = p \cdot s \\
Z'P \cdot Z'A' &= ZZ' \cdot Z'B' = p' \cdot s.
\end{aligned}
\tag{1}
$$

If now E' and E represent complex planes, then the relation of the points A' and A is that of a bilinear, or circular transformation:§

$$
z' = \frac{az + b}{cz + d}. \tag{2}
$$

3. *The line joining all pairs of corresponding points form a congruence* whose properties may be derived in the following manner: Pass a plane F through ZZ' and P, cutting the line of intersection e of E and E' in S and

* Presented to the American Mathematical Society, September, 1911.

† Steiner: Ges. Werke, vol. 1, pp. 407–439.

‡ Hirst: On Cremonian congruences. Proc. Lond. M. S., vol. 14, pp. 259–301. For further references see Sturm: Liniengeometrie, vol. 2.

§ Harkness & Morley: Introduction to Analytic Functions, pp. 42–44. Sturm: Die Lehre von den geometrischen Verwandtschaften, vol. IV, pp. 90–95.

the sphere in the circle C. In the inversions (Z, E) and (Z', E'), the straight lines SB and SB' respectively correspond to C. It is seen that the line AA' intersects the straight line through ZZ', and consequently, since P is any point on the sphere, all lines of the congruence pass through ZZ'. Letting P describe C and designating by P, P_1, P_2, $\cdots$ any set of points on

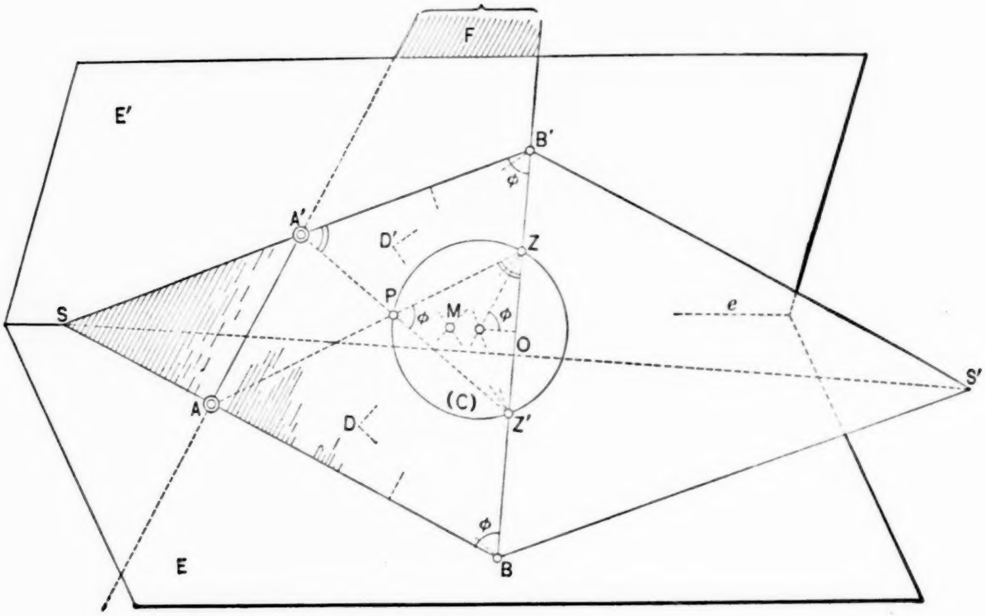

FIG. 1.

C and by A, A_1, A_2, $\cdots$ and A', A_1', A_2', $\cdots$ their projections on SB and SB' in E and E', there is plainly

$$(Z \cdot AA_1A_2 \cdots) \barwedge (Z' \cdot A'A_1'A_2' \cdots),$$

and consequently

$$(AA_1A_2 \cdots) \barwedge (A'A_1'A_2' \cdots). \tag{3}$$

The product of these projective ranges is a conic K. We have therefore the theorem:

The envelope of all lines of the congruence situated in a plane through ZZ' is a conic.

With every point P of the sphere, not coinciding with Z or Z', is associated a definite line AA' of the congruence. If P moves on C till it coincides with Z, it is plainly seen that, since the tangent to C at Z is parallel to SB, the ray corresponding to Z is $B'S' \parallel BS$. Similarly to Z' corresponds the line $BS' \parallel B'S$. From the figure $\Delta ZAB \sim \Delta ZZ'P$ and $\Delta ZZ'P \sim$

$\Delta A'Z'B'$, hence

$$\Delta ZAB \sim \Delta A'Z'B', \tag{4}$$

and $\sphericalangle ABZ = \sphericalangle A'B'Z'$. From this follows that BB' is equally inclined towards E and E' and that $SBS'B'$ is a rhombus. The conic K is therefore always in- or escribed to a rhombus and is accordingly an ellipse or an hyperbola with the middle point O of the distance BB' as a center. When the plane F turns about BB', the variable conic K in F generates a certain surface Q of which BB' is a singular line. The congruence of lines consists now of all lines passing through BB' and tangent to Q. From any point G of BB' two tangents can be drawn in every plane through BB', namely the tangents to the conic K in F. *The lines of the congruence through any point G of BB' form therefore a cone T of the second order.* Generally the order of a tangent cone to a surface is the same as that of the surface. As the singular line is a double line, the order of the complete tangent cone from G to Q is 4, *so that the surface Q itself is of the fourth order.* To determine order and class of the congruence, choose any point in space and pass a plane through it and BB'. The lines of the congruence in this plane envelope a conic and two of its tangents, or two lines of the congruence pass through the arbitrarily chosen point. The congruence is therefore of the second order. Any plane cuts BB' in a point G and the corresponding cone T in two elements. Every plane in a general position contains therefore two lines of the congruence, and the later is therefore of the second class. Hence the theorem:

The lines joining corresponding points in a circular transformation of two planes, as defined by (2), *form a* (2, 2) *congruence. All lines of the congruence are tangent to a surface Q of the fourth order and pass through the singular line of this surface.*

4. To determine the character of the surface Q more definitely, it is necessary to go back to the origin of the conic K. The directions of the axes of all conics in- or escribed to a rhombus coincide with the diagonals of the rhombus. The lengths of the axes of K in F, Fig. 1, are determined by those positions of AA' which are perpendicular to SS' and BB'. Designating the radius of C by ρ and the angle of BB' with BS or $B'S$ by φ, then from the figure

$$\cos\varphi = \frac{\sqrt{\rho^2 - \frac{s^2}{4}}}{\rho}, \qquad SB' = \frac{q}{2\cos\varphi} = \frac{q\rho}{\sqrt{4\rho^2 - s^2}},$$

$$SO = \sqrt{SB'^2 - \frac{q^2}{4}} = \sqrt{\frac{q^2\rho^2}{4\rho^2 - s^2} - \frac{q^2}{4}} = \sqrt{\frac{q^2 s^2}{4(4\rho^2 - s^2)}} = \frac{qs}{2\sqrt{4\rho^2 - s^2}}.$$

From (4) $AB \cdot A'B' = BZ \cdot B'Z' = pp'$. When $AA' \perp SS'$, $AB = A'B'$,

hence $AB^2 = pp'$. Designating in this case the perpendicular distance from O to AA' by α, then $\alpha^2 : AB^2 = SO^2 : SB'^2$, and, introducing the above values, solving for α^2 and reducing,

$$\alpha^2 = \frac{pp's^2}{4\rho^2}. \tag{5}$$

When $AA' \perp BB'$, then $AS + SB = AB$, $SB - SA = A'B'$, hence $SB^2 - AS^2 = pp'$, and $AS^2 = SB^2 - pp'$, or

$$AS^2 = \frac{q^2\rho^2}{4\rho^2 - s^2} - pp'.$$

Designating in this case by β the perpendicular distance of O from AA', we have $\beta^2 : AS^2 = OB^2 : SB^2$, and

$$\beta^2 = \frac{q^2\rho^2 - pp'(4\rho^2 - s^2).}{4\rho^2}. \tag{6}$$

By subtraction of (5) and (6) we get

$$\alpha^2 - \beta^2 = \frac{4pp' - q^2}{4} = k^2 \text{ (a constant)}. \tag{7}$$

From (5) we notice that α^2 is always a positive quantity, (p and p' both positive). We can however dispose in such a manner of the points Z, Z', B, B', that in (6) β^2 may be either positive, as in the case of the figure, or negative. In the first case ($\beta^2 > 0$) K is an ellipse, in the second case ($\beta^2 < 0$) K is an hyperbola. We may also have $k = 0$, in which case all conics K are circles. Hence from (7) the result:

The focal distance of all conics K is constant, i. e., the foci of all conics K are situated on a circle whose plane is perpendicular to BB' and whose center is at O.

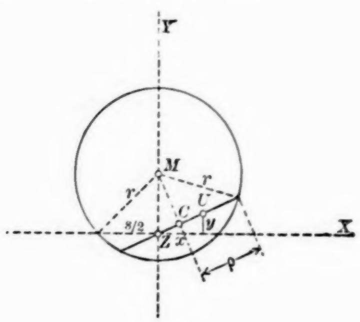

FIG. 2.

To establish the equation of the surface Q, choose BB' as the z-axis, the lines through O parallel and perpendicular to e as the x- and y-axes. Projecting the configuration on the xy-plane, Fig. 2 is obtained. The sphere appears as a circle, and the plane F with the conic K as a chord through Z. Let x, y, z be the coördinates of any point U on K (surface Q), then from Fig. 2,

$$\frac{y^2}{x^2} = \frac{CZ^2}{CM^2} = \frac{r^2 - \dfrac{s^2}{4} - r^2 + \rho^2}{r^2 - \rho^2}$$

and from this

$$\rho^2 = \frac{s^2x^2 + 4r^2y^2}{4(x^2 + y^2)}.$$

The equation of the surface Q is now

$$\frac{x^2 + y^2}{\alpha^2} + \frac{z^2}{\beta^2} = 1,$$

or, after replacing α^2, β^2 by their values in (5) and (6) and also ρ^2 by the above expression, and reducing,

$$\begin{aligned}(8)\qquad &(x^2 + y^2)[pp's^2(s^2x^2 + 4r^2y^2) - p^2p'^2s^4] - (s^2x^2 + 4r^2y^2)\\ &\qquad [k^2(s^2x^2 + 4r^2y^2) - pp's^2z^2 - k^2pp's^2] = 0,\end{aligned}$$

which clearly is of the fourth degree.

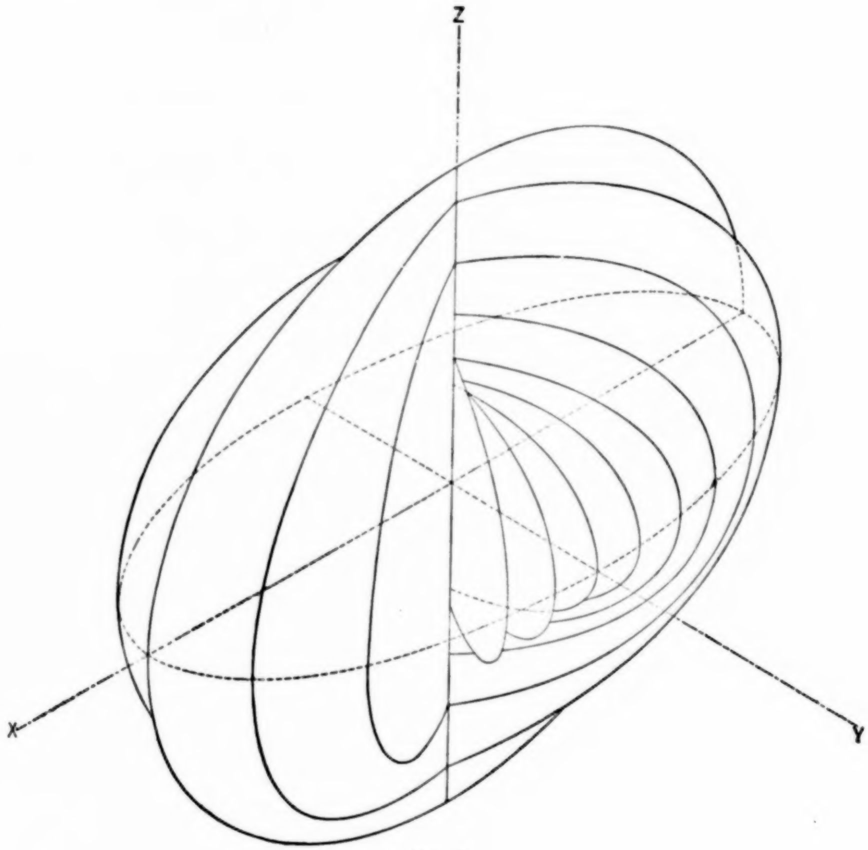

Fig. 3.

The intersection of Q with the xy-plane degenerates into the ellipse

$$\frac{x^2}{pp'} + \frac{y^2}{\frac{pp's^2}{4r^2}} = 1$$

and the point $(x = 0, y = 0)$.

5. The peculiar character of this surface is shown in Fig. 3 by an isometric projection. In this representation $r = s = 1$, $p = 6$, $p' = 5$, $q = p + p' - s = 10$, $k^2 = 5$; and the principal axis of Q along the x-axis measures 10 units.

The equation becomes

$$(9) \qquad (x^2 + 4y^2 - 30)(5x^2 + 2y^2) + (x^2 + 4y^2)6z^2 = 0$$

under these assumptions.

6. As stated before, when $k^2 = 4pp' - q^2 = 0$, all conics K become circles. The condition is equivalent to $4pp' - (p + p' - s)^2 = 0$, or $(p - p')^2 - 2(p + p')s + s^2 = 0$. This gives for s the values $s = (\sqrt{p} \pm \sqrt{p'})^2$. The equation of Q reduces to

$$(10) \qquad (x^2 + y^2 + z^2)(s^2x^2 + 4r^2y^2) - pp's^2(x^2 + y^2) = 0.$$

For $p = g$, $p' = 4$ we get as a value of s, $s = 1$, and putting $r = 1$, (10) becomes

$$(11) \qquad (x^2 + y^2 + z^2)(x^2 + 4y^2) - 36(x^2 + y^2) = 0.$$

University of Illinois.

ON DUHAMEL'S THEOREM.

By R. L. Moore.

In these *Annals*, July, 1903, pages 161–178, Osgood discusses the following proposition which he refers to Duhamel:

Proposition I. "Let $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ be a sum of positive infinitesimals which approaches a limit when $n = \infty$. Let $\beta_1 + \beta_2 + \cdots + \beta_n$ be a second sum of infinitesimals which differ from the infinitesimals of the first sum by infinitesimals of higher order; *i. e.*, let

$$\operatorname{Lim} \frac{\beta_i}{\alpha_i} = 1.$$

Then the second sum approaches a limit when $n = \infty$, and this limit is the same as that of the first sum:

$$\operatorname*{Lim}_{n=\infty} (\beta_1 + \beta_2 + \cdots + \beta_n) = \operatorname*{Lim}_{n=\infty} (\alpha_1 + \alpha_2 + \cdots + \alpha_n)."$$

Osgood calls attention to two objections to Proposition I, as it stands. First, whether it is true or not depends on what interpretation is given to the requirement that $\operatorname{Lim} \beta_i/\alpha_i = 1$. Secondly, there are certain simple problems where the hypothesis of I is not completely satisfied though the conclusion holds. After discussing these points he formulates a substitute theorem* which is at once rigorously true and well adapted to application in the problems above mentioned.

In the present paper I propose a second form,† which is suitable for application to a still wider range of problems and at the same time seems to be even easier‡ to apply.

In many, if not all, applications of this theorem the numbers α_{in}§ and β_{in}§ are related to subdivisions of an interval or (more generally) to sub-sets of a limited point-set in m-dimensional space. Let E be such a point-set. For each value of the positive integer n let $E_{1n}, E_{2n}, \cdots, E_{nn}$, be non-overlapping sub-sets of E of interior measures $e_{1n}, e_{2n}, \cdots, e_{nn}$, respectively. Let us now consider the numbers α_{in} and β_{in} as being related (through the

* Loc. cit., page 173.

† Theorem 2, page 162.

‡ This statement is subject to certain reservations which are indicated in a footnote on page 165.

§ I prefer to write $\alpha_{1n}, \alpha_{2n} \cdots \alpha_{nn}$ instead of $\alpha_1, \alpha_2 \cdots \alpha_n$. Double subscripts are evidently desirable.

intermediation of their subscripts)* to E_{in} for each pair of values of n and i $(i = 1, \cdots, n)$.

Osgood suggests† that, in certain problems at least, a natural interpretation of the requirement that

$$\lim_{n=\infty} \frac{\beta_{in}}{\alpha_{in}} = 1$$

might be‡ that, for every point P of the point-set E,

$$\lim_{n=\infty} \frac{\beta_{i_{P_n}n}}{\alpha_{i_{P_n}n}} = 1,$$

where i_{P_n} *signifies such a value of* i *that* $E_{i_{P_n}n}$ *shall contain the point* P.

But he gives§ an independence example to show that, under this interpretation, I is not a true proposition. If one examines this independence example, however, it is to be observed that here there is no upper bound to the set of values of the ratio

$$\left|\frac{\beta_{in}}{\alpha_{in}}\right|.$$

I have found that if to the hypothesis of I there be added the requirement that such a *bound shall exist* and

$$\lim_{n=\infty} \frac{\beta_i}{\alpha_i} = 1$$

be given the *interpretation suggested above, I is thereby transformed into a true proposition.* Moreover this proposition remains true even if a set of points (P) of measure 0 be relieved from the requirement that

$$\lim_{n=\infty} \frac{\beta_{i_{P_n}n}}{\alpha_{i_{P_n}n}} = 1.$$

The following theorem holds true.

Theorem 2. HYPOTHESIS:‖ (*a*) *E is a limited point-set in a space of m-dimensions. $E_{1n}, E_{2n}, \cdots, E_{nn}$, are (for each value of the positive integer n) non-overlapping sub-sets of E of interior measures $e_{1n}, e_{2n}, \cdots, e_{nn}$, respectively. r_{in}, r_{in}' $(i = 1, \cdots, n)$, are numbers such that the set $\{|r_{in}' - r_{in}|\}$ is a bounded set, i. e., there exists a number c such that for all values of n and i*

* Thus, for example, α_{12} and β_{12} correspond to E_{12}; α_{45} and β_{45} correspond to E_{45}.

† Loc. cit., page 170.

‡ Aside from phraseology, notation, etc.

§ Loc. cit., pages 170 and 171.

‖ In close parallelism with Osgood I write $r_{in}e_{in}$, $r_{in}'e_{in}'$ instead of α_{in}, β_{in} and avoid the division of β_{in} by α_{in}. There is an evident advantage in this in cases where α_{in} is sometimes 0.

$(i \leqq n)$, $|r_{in}' - r_{in}| \leqq c$. (b) $Lim_{n=\infty} \sum_{i=1}^{n} r_{in}e_{in}$ exists. (c) E_0 is a subset of E of measure 0. (d) If P is a point of E not belonging to E_0 then

$$\underset{n=\infty}{Lim}\,(r_{i_{P_n}n}' - r_{i_{P_n}n}) = 0.$$

CONCLUSION: *$Lim_{n=\infty} \sum_{i=1}^{n} r_{in}'e_{in}$ exists and equals $Lim_{n=\infty} \sum_{i=1}^{n} r_{in}e_{in}$.*

PROOF. If $\epsilon > 0$ and $\delta > 0$, there exists $n_{\epsilon\delta}$ such that if $n > n_{\epsilon\delta}$ then $S_n \leqq \epsilon$, where S_n denotes the sum of the interior measures of those of the point-sets $E_{1n}, E_{2n}, \cdots, E_{nn}$, for which the corresponding* $|r_{in}' - r_{in}| > \delta$. For otherwise there would be some value ϵ and some infinite sequence of values of n such that, for each value of n belonging to this sequence, $S_n > \epsilon$. Therefore, according to a theorem of W. H. Young's,† generalized so as to apply to space of any finite number of dimensions, there would exist a point set K of interior measure greater than or equal to ϵ, each point of K being contained, for an infinite sequence of different values of n, in some E_{in} for which the corresponding $|r_{in}' - r_{in}| > \delta$. But this would be contrary to Hypothesis (*d*), according to which K must be of measure 0.

It is true, then, that if $\epsilon > 0$ and $\delta > 0$ there exists $n_{\epsilon\delta}$ such that if $n > n_{\epsilon\delta}$ then $S_n \leqq \epsilon$. Hence if $n > n_{\epsilon\delta}$ then

$$\left| \sum_{i=1}^{n} (r_{in} - r_{in}')e_{in} \right| \leqq \epsilon c + \delta e',$$

where e' is the exterior measure of E. Of course by proper choice of ϵ and δ the right hand member of this inequality may be made less than any preassigned positive number. Hence, in view of Hypothesis (*b*), it follows that $\mathrm{Lim}_{n=\infty} \sum_{i=1}^{n} r_{in}'e_{in}$ exists and is equal to $\mathrm{Lim}_{n=\infty} \sum_{i=1}^{n} r_{in}e_{in}$.

Osgood's Theorem‡ is as follows:

Osgood's Theorem. *Let*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n \qquad (A)$$

be a sum of infinitesimals and let α_i differ uniformly by an infinitesimal of higher order than Δx_i from the summand $f(x_i)\Delta x_i$ of the definite integral

$$\int_a^b f(x)dx \qquad (B)$$

of the function $f(x)$, this function being continuous throughout the interval $a \leqq x \leqq b$. Then the sum (A) approaches a limit when $n = \infty$, and the value of this limit is the definite integral (B):

$$\underset{n=\infty}{\mathrm{Lim}} \sum_{i=1}^{n} \alpha_i = \int_a^b f(x)dx.$$

* $r_{i_1 n}$ is said to correspond to $E_{i_2 n}$ if $i_1 = i_2$.

† W. H. Young, Lond. Math. Soc. Proc., Ser. 2, vol. II, p. 25.

‡ Loc. cit., page 173.

With regard to ease of application, Theorem 2 has an advantage over Osgood's Theorem in that it is not necessary in applying Theorem 2 to prove that $\mathrm{Lim}_{n=\infty}(r_{in}' - r_{in}) = 0$ *uniformly*.*

An Application.—Let us consider a problem to which Osgood applies† his Theorem, the problem of finding the attraction of a material body on a particle lying without it. Following Osgood, let the coördinates of the particle P be (a, b, c), let its mass be m, let V be the volume of the material body E and let X be the component along OX of the attraction which the body E exerts on the particle P. Divide the body E into n small pieces E_{1n}, $E_{2n}, \cdots, E_{nn}$ of volumes $\Delta V_{1n}, \Delta V_{2n}, \cdots, \Delta V_{nn}$, respectively, where as $n \doteq \infty$ the maximum diameter of E_{in} approaches 0 as a limit. With‡ Osgood "Let ρ_{in}'', γ_{in}'', α_{in}'' denote respectively the maximum values of the density ρ in E_{in}, the distance γ from (a, b, c) to a point of E_{in}, and the angle α which a ray drawn from (a, b, c) to a point of E_{in} makes with the positive axis of x; and let ρ_{in}', γ_{in}', α_{in}' denote respectively the minimum values of these functions." Osgood derives the double inequality: §

$$k\frac{m\rho_{in}'\Delta V_{in}}{\gamma_{in}''^2}\cos\alpha_{in}'' < \Delta X_{in} < k\frac{m\rho_i''\Delta V_{in}}{\gamma_{in}'^2}\cos\alpha_{in}', \tag{4}$$

where k is a constant of proportionality.

From this point I would proceed as follows:

Let (4)′ designate the double inequality obtained by dividing (4) by ΔV_{in}. If P is any point of E then as $n \doteq \infty$ all points of $E_{i_{P_n}n}$ approach P as a limit. Hence, since ρ, γ and $\cos\alpha$ are continuous functions of (x, y, z) and γ is never 0 in E by hypothesis, therefore

$$\mathrm{Lim}_{n=\infty}\frac{km\rho_{i_{P_n}n}'\cos\alpha_{i_{P_n}n}''}{\gamma_{i_{P_n}n}''^2} = \mathrm{Lim}_{n=\infty}\frac{km\rho_{i_{P_n}n}''\cos\alpha_{i_{P_n}n}'}{\gamma_{i_{P_n}n}'^2} = \mathrm{Lim}_{n=\infty}\frac{km\rho_{i_{P_n}n}\cos\alpha_{i_{P_n}n}}{\gamma_{i_{P_n}n}^2},$$

where $\theta_{i_{P_n}n}$, $\cos\alpha_{i_{P_n}n}$, $\gamma_{i_{P_n}n}$ apply to any arbitrarily chosen point of $E_{i_{P_n}n}$. Hence, from (4)′,

$$\mathrm{Lim}_{n=\infty}\frac{\Delta X_{i_{P_n}n}}{\Delta V_{i_{P_n}n}} = \mathrm{Lim}_{n=\infty}\frac{km\rho_{i_{P_n}n}\cos\alpha_{i_{P_n}n}}{\gamma_{i_{P_n}n}^2}.$$

Clearly the left hand member and the right hand member of (4)′ are bounded. Hence the middle term, $\Delta X_{in}/\Delta V_{in}$, is bounded. It follows, then, from

* Cf. uniformity condition in Osgood's Hypothesis.

† Loc. cit., pp. 166–169 and 174–175.

‡ Except for double subscripts and other slight changes.

§ Loc. cit., page 168, double inequality (4).

Theorem 2, that

$$\operatorname*{Lim}_{n=\infty}\sum_{i=1}^{n}\Delta X_{in} = \operatorname*{Lim}_{n=\infty}\sum_{i=1}^{n}\frac{km\rho_{in}\cos\alpha_{in}}{\gamma_{in}^{2}}\Delta V_{in},$$

which is equal to

$$\int_V\int\int k\frac{m\rho\cos\alpha}{\gamma^2}dV.$$

Compare this argument with Osgood's argument on page 174, loc. cit.

In this problem if the density ρ should be discontinuous at a set of points E_0 of measure 0 (and continuous elsewhere throughout E) then in the above argument merely stipulate that the point P shall not belong to E_0.

Further Remarks.—The remarks in this section apply to those cases in which as $n \doteq \infty$ the distance between each pair of points that belong to the same E_{in} approaches 0, E is a closed set of points and for each value of n each point of E belongs to some E_{in}.

I wish to discuss three conditions which I will call A, B and C.

A: If P is any point of E then

$$\operatorname*{Lim}_{n=\infty}(r_{i_{P_n}n}' - r_{i_{P_n}n}) = 0$$

B: $\operatorname*{Lim}_{n=\infty}(r_{i_{P_n}n}' - r_{i_{P_n}n}) = 0$ uniformly as to i.

C: If P is any point of E then $\operatorname*{Lim}_{n=\infty}(r_{in}' - r_{in}) = 0$ for all such modes of variation of i with n that, as $n \doteq \infty$, some point of E_{in} approaches* P as its limit.

Let A' and C' denote the conditions obtained by substituting $E - E_0$ for E in conditions A and C respectively, where E_0 is a sub-set of E of measure 0.

As has been stated, A is less exacting than B, which corresponds to Osgood's uniformity condition. It may be easily seen however that B is equivalent to C. It is of interest, then, to ascertain whether A' (which I use in Theorem 2) is as exacting as C' which is a condition obtained by weakening (an equivalent of) the uniformity condition B by exempting a set of points (P) of measure 0 from its jurisdiction. The following independence example shows that A' is less exacting than even C'.

* If the word "is" is substituted here for the word "approaches" then C becomes identical with A. In, for instance, the above problem in mechanics where the density is a continuous function of (x, y, z), A and C could be verified with equal or almost equal facility. But C is equivalent to B which corresponds to Osgood's uniformity condition. These facts are to be considered if one compares Osgood's theorem and mine with regard to ease of application in those cases where no discontinuities enter.

Independence Example.—Assume that the rational points of the interval (0, 1) have, by some scheme, been brought into a one-to-one correspondence with the set of all positive integers. Let t_k denote that rational number which, according to this scheme, corresponds to the positive integer k. Divide the interval (0, 1) into n equal sub-intervals E_{in} by the points x_{in} where $i = 1 \cdots n$. Let $r_{in}' = 1$, only when $n = 2^k$ where k is such a positive integer that $x_{(i-1)n} \leqq t_k \leqq x_{in}$. In all other cases let $r_{in}' = 0$. Ler $r_{in} = 0$ for all values of n and i. Then it may be shown that if P is any point in (0, 1) then in every neighborhood of P there is an interval $E_{i_1n_1}$, such that $r_{i_1n_1}' - r_{i_1n_1} = 0$ and, at the same time, another interval $E_{i_2n_2}'$ such that $r_{i_2n_2}' - r_{i_2n_2} = 1$. Thus condition C is satisfied with respect to *no point whatsoever* of the interval (0, 1). On the other hand condition A' is satisfied with respect to *every point* of that interval.

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ON LINEAR EQUATIONS WITH AN INFINITE NUMBER OF VARIABLES.

By Maxime Bôcher and Louis Brand.

E. Schmidt's treatment of a system of linear equations with an infinite number of variables* is of such essential simplicity and importance that it seems destined to become classical. The original memoir, however, owing to its condensation and to the rather abstract form which it has in parts is not entirely easy reading for the beginner, and Kowalewski's presentation,† while attractive in some respects, is extremely long and so arranged that unless one reads the whole it is almost impossible to get at the essential results.

The following treatment, which so far as it goes is complete in itself, is a modification of those heretofore given. Its characteristic features are, on the one hand, that it avoids altogether the process of normalization which plays such an essential and often repeated rôle in the earlier treatments; and, on the other hand, that it deals first with the case of a finite number of equations involving an infinite number of variables and regards the case of an infinite number of equations as a limit.

For the sake of clearness, though this is not logically necessary, the algebraic case of a finite number of variables is taken up first.

1. Complex Quantities with k Components. The real and complex quantities of ordinary algebra shall be termed *scalars* in distinction to the higher complex quantities, $(a_1, a_2, \cdots, a_k)$, which are aggregates of k scalars —the *components* of the complex quantity—taken in a definite order. Such complex quantities will be denoted by Greek letters. That complex quantity whose components are all zero shall be denoted by 0. Two complex quantities,

$$\alpha = (a_1, a_2, \cdots, a_k), \quad \beta = (b_1, b_2, \cdots, b_k),$$

are said to be equal when and only when $a_i = b_i\,(i = 1, 2, \cdots, k)$. We define the sum of α and β by

$$\alpha + \beta \equiv (a_1 + b_1, a_2 + b_2, \cdots, a_k + b_k);$$

and the product of α by a scalar, p, by

$$p\alpha \equiv \alpha p \equiv (pa_1, pa_2, \cdots, pa_k).$$

* Rendiconti del Circolo Matematico di Palermo, vol. 25 (1908), pp. 56–77.

† Einführung in die Determinantentheorie (Veit: Leipzig, 1909), pp. 407–455.

The n complex quantities, $\alpha_1, \alpha_2, \cdots, \alpha_n$ are said to be linearly dependent if there exist n scalars, $c_1, c_2, \cdots, c_n$, not all zero, such that

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0.$$

In view of the definition of the complex quantity 0 this is equivalent to saying that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly dependent when and only when the n sets of k scalars each forming their components are linearly dependent. Any $k + 1$ complex quantities having k components are therefore linearly dependent.* When less than k complex quantities are given, there are always others linearly independent of them.

We also consider the *inner product*, or simply *product*, of two complex quantities α and β, defined to be the scalar

$$\alpha\beta \equiv a_1b_1 + a_2b_2 + \cdots + a_kb_k.$$

We note that $\alpha\beta$ may vanish when $\alpha \neq 0$, $\beta \neq 0$. From this definition it is clear that the commutative and distributive laws,

$$\alpha\beta = \beta\alpha, \qquad \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma,$$

and the associative law in the case of multiplication by a scalar p,

$$p(\alpha\beta) = (p\alpha)\beta = \alpha(p\beta),$$

all hold good. The associative law, in the case of the product of three or more complex quantities, is not true. Thus $\alpha\beta\gamma$ is meaningless unless either $(\alpha\beta)\gamma$ or $\alpha(\beta\gamma)$ is specified.

A dash above a scalar shall denote, as usual, its conjugate imaginary scalar; and we shall extend this notation by writing

$$\bar{\alpha} \equiv (\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_k).$$

Then

$$\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}.$$

By the *norm* of the complex quantity α is understood the scalar

$$\text{norm } \alpha \equiv \alpha\bar{\alpha} = a_1\bar{a}_1 + a_2\bar{a}_2 + \cdots + a_k\bar{a}_k = |a_1|^2 + |a_2|^2 + \cdots + |a_k|^2,$$

which is always real. Clearly norm α = norm $\bar{\alpha}$. Norm α is 0 when and only when $\alpha = 0$, and is otherwise positive.

2. Homogeneous Linear Algebraic Equations. Consider now a system of n homogeneous equations in k unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kk}x_k = 0. \end{cases}$$

* See, for example, Bôcher's Higher Algebra, § 13.

We may regard the coefficients of each of these equations as the components of a complex quantity:

$$\alpha_i = (a_{i1}, a_{i2}, \cdots, a_{ik}) \qquad (i = 1, 2, \cdots, n),$$

and also the x's as the components of the complex quantity

$$\xi = (x_1, x_2, \cdots, x_k).$$

Our system of equations may then be written

$$(1) \qquad \alpha_1\xi = 0, \alpha_2\xi = 0, \cdots, \alpha_n\xi = 0.$$

THEOREM 1. *If ξ satisfies equations* (1) *and is linearly dependent upon $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, then $\xi = 0$.*

For suppose that

$$\xi = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_n\bar{\alpha}_n.$$

Then multiplying equations (1) by $\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_n$ respectively and adding we get

$$(\bar{c}_1\alpha_1 + \bar{c}_2\alpha_2 + \cdots + \bar{c}_n\alpha_n)\xi = \bar{\xi}\xi = 0.$$

Hence $\xi = 0$, as was to be proved.

COROLLARY. If ξ satisfies the equations

$$\bar{\alpha}_1\xi = 0, \bar{\alpha}_2\xi = 0, \cdots, \bar{\alpha}_n\xi = 0$$

and is linearly dependent upon $\alpha_1, \alpha_2, \cdots, \alpha_n$, then $\xi = 0$.

We are now in position to obtain a criterion for the linear dependence of n complex quantities. If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly dependent,

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0,$$

where not all the c's are zero. Multiplying this relation in succession by $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, we obtain the n equations

$$c_1\alpha_1\bar{\alpha}_i + c_2\alpha_2\bar{\alpha}_i + \cdots + c_n\alpha_n\bar{\alpha}_i = 0 \quad (i = 1, 2, \cdots, n).$$

In this system of homogeneous, linear equations in $c_1, c_2, \cdots, c_n$ the c's are not all zero and hence the determinant of the system must vanish. We call this determinant, which it should be noticed is a *real* scalar, the *Gramian* of $\alpha_1, \alpha_2, \cdots, \alpha_n$ and denote it by $G(\alpha_1, \alpha_2, \cdots, \alpha_n)$. Thus

$$(2) \qquad G(\alpha_1, \alpha_2, \cdots, \alpha_n) = \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n \\ \alpha_2\bar{\alpha}_1 & \alpha_2\bar{\alpha}_2 & \cdots & \alpha_2\bar{\alpha}_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n \end{vmatrix}.$$

The relation $G = 0$ is therefore a necessary condition for linear dependence.

It is also sufficient. For suppose that $G = 0$; then the n sets of scalars forming the rows of the Gramian are linearly dependent, and we have

$$\bar{\alpha}_i(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) = 0 \qquad (i = 1, 2, \cdots, n),$$

where not all of the c's vanish. We now infer from the Corollary of Theorem 1 that

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0,$$

which establishes the linear dependence of $\alpha_1, \alpha_2, \cdots, \alpha_n$. We have thus proved

THEOREM 2. *A necessary and sufficient condition that the complex quantities $\alpha_1, \alpha_2, \cdots, \alpha_n$ be linearly dependent is that their Gramian vanish.**

We turn now to the solution of the system (1), assuming that these equations are linearly independent, so that $G(\alpha_1, \alpha_2, \cdots, \alpha_n) \neq 0$. Every complex quantity, and therefore every solution ξ_1 of (1), can be written in the form

$$\xi_1 = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_n\bar{\alpha}_n + \bar{\eta} \tag{3}$$

where η is some complex quantity. In order that this be a solution of (1), the scalars c_i must satisfy the n relations

$$\left\{\begin{array}{l} c_1\alpha_1\bar{\alpha}_1 + c_2\alpha_1\bar{\alpha}_2 + \cdots + c_n\alpha_1\bar{\alpha}_n = -\alpha_1\bar{\eta} \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ c_1\alpha_n\bar{\alpha}_1 + c_2\alpha_n\bar{\alpha}_2 + \cdots + c_n\alpha_n\bar{\alpha}_n = -\alpha_n\bar{\eta}. \end{array}\right. \tag{4}$$

Solving these equations for the c's and substituting in (3), we have

$$\xi_1 = \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & \alpha_1\bar{\eta} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & \alpha_n\bar{\eta} \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n & \bar{\eta} \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{5}$$

Every solution of (1) can therefore be expressed in this form. That, conversely, no matter what the complex quantity η may be, the expression (5) always gives a solution of (1) is seen at once by direct substitution; for if we form the product $\alpha_i\xi_1$ by multiplying the last row of the determinant in the numerator by α_i, this row becomes identical with the ith row.

* We note in passing that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are connected by the *same* linear relation that connects the rows of their Gramian, written as above.

THEOREM 3. *If the equations*

$$\alpha_1\xi = 0,\ \alpha_2\xi = 0,\ \cdots,\ \alpha_n\xi = 0,$$

are linearly independent, their general solution is given by (5), *where* η *is an arbitrary complex quantity.*

When ξ_1 vanishes, we see from (3) that $\bar{\eta}$ is linearly dependent upon $\bar{\alpha}_1$, $\bar{\alpha}_2, \cdots, \bar{\alpha}_n$. Conversely, if $\bar{\eta}$ is linearly dependent upon $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, the same is true of ξ_1, and hence, by Theorem 1, $\xi_1 = 0$. Now to two $\bar{\eta}$'s correspond two ξ_1's whose difference is precisely that solution of (1) which corresponds to the difference between the $\bar{\eta}$'s. Consequently two different $\bar{\eta}$'s yield the same ξ_1 when and only when their difference is linearly dependent upon $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$.

If $n > k$ the equations (1) are necessarily linearly dependent, so that Theorem 3 does not apply to this case. If $n = k$ every $\bar{\eta}$ is linearly dependent on the $\bar{\alpha}$'s, so that in this case, as is well known, equations (1) have only the trivial solution zero. If $n < k$ we can find $k - n$ complex quantities $\bar{\alpha}_{n+1}, \bar{\alpha}_{n+2}, \cdots, \bar{\alpha}_k$ such that $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_k$ are linearly independent. Then every η may be written as $C_1\bar{\alpha}_1 + C_2\bar{\alpha}_2 + \cdots + C_k\bar{\alpha}_k$; but as a change in $\bar{\eta}$ by a quantity linearly dependent upon $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$ does not affect formula (5), we lose nothing in generality if we assume $\bar{\eta}$ of the form

$$\bar{\eta} = C_{n+1}\bar{\alpha}_{n+1} + \cdots + C_k\bar{\alpha}_k.$$

Thus the solution (5) contains, as it should, $k - n$ arbitrary scalars, C_{n+1}, $\cdots, C_k$, and contains them linearly and homogeneously.

A formula for the norm of ξ_1 is readily found. From (3):

$$\text{(6)} \qquad \text{norm } \xi_1 = \bar{c}_1\alpha_1\xi_1 + \cdots + \bar{c}_n\alpha_n\xi_1 + \eta\xi_1 = \eta\xi_1.$$

If we form the product $\eta\xi_1$ from (5) by multiplying the last row of the determinant in the numerator by η, it is clear that

$$\text{(7)} \qquad \text{norm } \xi_1 = \frac{G(\alpha_1, \alpha_2, \cdots, \alpha_n, \eta)}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}.$$

We proceed to use this relation to establish an important property of Gramians. In (7) $\alpha_1, \alpha_2, \cdots, \alpha_n, \eta$ may be regarded as $n+1$ arbitrary complex quantities; we will assume that they are linearly independent. Then $\bar{\eta}$ is clearly not a linear combination of $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, so that $\xi_1 \neq 0$ and norm $\xi_1 > 0$. Moreover this assumption entails that none of $\alpha_1, \alpha_2, \cdots$, α_n, η vanish, and hence the Gramian of any one, *e. g.*, $G(\alpha_1) = \alpha_1\bar{\alpha}_1$, is real and positive. Hence by giving to n in (7) in succession the values $1, 2, \cdots$, we establish by mathematical induction

THEOREM 4. *The Gramian of any number of linearly independent complex quantities is real and positive.*

3. Non-Homogeneous Linear Algebraic Equations. We come now to the system of non-homogeneous equations

$$\alpha_1\xi = b_1,\ \alpha_2\xi = b_2,\ \cdots,\ \alpha_n\xi = b_n, \tag{8}$$

where we again assume that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent, and try to find a solution of the form

$$\xi_0 = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 + \cdots + c_n\bar{\alpha}_n. \tag{9}$$

Substituting this in (8), we obtain n linear equations, which may be obtained from equations (4) by replacing their right hand members by $b_1, b_2, \cdots, b_n$ respectively. These can, as above, be solved for the c's by Cramer's rule, and the results substituted in (9). This gives

$$\xi_0 = \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & -b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & -b_n \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n & 0 \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{10}$$

That this is really a solution of (8) we see by direct substitution. For if we form the product $\alpha_i\xi_0$, the last row of the determinant in the numerator becomes

$$\alpha_i\bar{\alpha}_1,\ \alpha_i\bar{\alpha}_2,\ \cdots,\ \alpha_i\bar{\alpha}_n,\ 0;$$

and, when the ith row is subtracted from this, it appears that

$$\alpha_i\xi_0 = b_iG/G = b_i.$$

We have thus proved

THEOREM 5. *If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent, the equations* (8) *have one and only one solution of the form* (9), *and this is given by* (10).

The general solution of (8) is of course obtained by adding to the particular solution (10) the general solution (5) of the homogeneous equations (1); it is therefore

$$\xi = \xi_0 + \xi_1 = \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & \alpha_1\bar{\eta} - b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & \alpha_n\bar{\eta} - b_n \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n & \bar{\eta} \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{11}$$

The solution (10) of (8), which is characterized by being the only solution of (8) which is linearly dependent upon the $\bar{\alpha}$'s, shall be called the *principal solution* of (8). It has also another characteristic property which may

be deduced as follows. From (11) we see that

$$\xi\bar{\xi} = (\xi_0 + \xi_1)(\bar{\xi}_0 + \bar{\xi}_1) = \xi_0\bar{\xi}_0 + \xi_0\bar{\xi}_1 + \xi_1\bar{\xi}_0 + \xi_1\bar{\xi}_1;$$

and from (9)

$$\xi_0\bar{\xi}_1 = c_1\bar{\alpha}_1\bar{\xi}_1 + c_2\bar{\alpha}_2\bar{\xi}_1 + \cdots + c_n\bar{\alpha}_n\bar{\xi}_1 = 0,$$

remembering that ξ_1 is a solution of equations (1). Consequently $\xi_1\bar{\xi}_0 = 0$, and

$$\text{norm } \xi = \text{norm } \xi_0 + \text{norm } \xi_1, \tag{12}$$

so that

$$\text{norm } \xi \geqq \text{norm } \xi_0,$$

the equality sign holding only when $\xi_1 = 0$, in which case $\xi = \xi_0$. Thus we have

THEOREM 6. *Among the solutions of* (8) *no other has so small a norm as the principal solution.*

To obtain a formula for norm ξ_0 we multiply the last row of the determinant in the numerator of (10) by $\bar{\xi}_0$ and simplify by use of the equations $\bar{\alpha}_i\xi_0 = \bar{b}_i$; thus*

$$\text{norm } \xi_0 = -\frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & b_n \\ \bar{b}_1 & \bar{b}_2 & \cdots & \bar{b}_n & 0 \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{13}$$

Norm ξ is now given by (12).

4. System of a Finite Number of Linear Equations in an Infinite Number of Variables. We now consider a system of n equations

$$a_{i1}x_1 + a_{i2}x_2 + \cdots = 0 \qquad (i = 1, 2, \cdots, n), \tag{14}$$

where the number of unknowns $x_1, x_2, \cdots$ is infinite. For this purpose we use complex quantities with an infinite number of components. If $\alpha = (a_1, a_2, \cdots)$ is such a complex quantity, we consider the series $|a_1|^2 + |a_2|^2 + \cdots$. If this series is convergent, we say that the complex quantity has a finite norm and define

$$\text{norm } \alpha \equiv |a_1|^2 + |a_2|^2 + \cdots, \quad |\alpha| \equiv \sqrt{\text{norm } \alpha}.$$

* If not all of the b's vanish, it is clear from equations (8) that $\xi_0 \neq 0$, and hence norm $\xi_0 > 0$. By means of (13) we may now prove at once the following

THEOREM. *If the Gramian of linearly independent complex quantities is bordered by scalars that do not all vanish so as to form a determinant of the type of that in* (13), *this bordered Gramian is negative.*

The sum of $\alpha = (a_1, a_2, \cdots)$ and $\beta = (b_1, b_2, \cdots)$, and the product of α by a scalar p are defined as

$$\alpha + \beta \equiv (a_1 + b_1, a_2 + b_2, \cdots), \quad p\alpha \equiv \alpha p \equiv (pa_1, pa_2, \cdots).$$

The product $\alpha\beta$ we define by the formula

$$\alpha\beta \equiv a_1b_1 + a_2b_2 + \cdots$$

whenever this series converges. When α and β have finite norms their product $\alpha\beta$ always exists, as then the series in question is absolutely convergent. For writing

$$\alpha_k = (|a_1|, |a_2|, \cdots, |a_k|), \quad \beta_k = (|b_1|, |b_2|, \cdots, |b_k|)$$

we have from Theorems 2 and 4

$$G(\alpha_k, \beta_k) = \begin{vmatrix} \alpha_k\bar{\alpha}_k & \alpha_k\bar{\beta}_k \\ \beta_k\bar{\alpha}_k & \beta_k\bar{\beta}_k \end{vmatrix} \geqq 0.$$

Hence, as $\alpha_k = \bar{\alpha}_k$ and $\beta_k = \bar{\beta}_k$,

$$(\alpha_k\beta_k)^2 \leqq \text{norm } \alpha \cdot \text{norm } \beta$$

or

$$|a_1b_1| + \cdots + |a_kb_k| \leqq |\alpha| \, |\beta|.$$

Since this holds for all values of k, the absolute convergence of our series is established.

The distributive law, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, evidently holds when $\alpha\beta$ and $\alpha\gamma$ have meanings. Thus, in particular, if α and β have finite norms, we have

$$\text{norm } (\alpha + \beta) = (\alpha + \beta)(\alpha + \bar{\beta}) = \alpha\bar{\alpha} + \beta\bar{\alpha} + \alpha\bar{\beta} + \beta\bar{\beta},$$

so that if two complex quantities have finite norms their sum also has a finite norm. It is also obviously true that if a complex quantity has a finite norm it will still have a finite norm after being multiplied by a scalar. From these two facts we readily infer that if a number of complex quantities have finite norms any complex quantity linearly dependent upon them also has a finite norm.

Using the $n + 1$ complex quantities

$$\alpha_i = (a_{i1}, a_{i2}, \cdots) \qquad (i = 1, 2, \cdots, n),$$
$$\xi = (x_1, x_2, \cdots),$$

the equations (14) may be written

$$\alpha_1\xi = 0, \; \alpha_2\xi = 0, \; \cdots, \; \alpha_n\xi = 0. \tag{15}$$

We place upon the coefficients α_i the restriction that they have finite norms. Then ξ is to be so determined that the series $\alpha_i\xi$ all converge to the value

zero. If ξ has a finite norm the series $\alpha_i\xi$ necessarily converge, but this may also be the case when ξ has an infinite norm.

THEOREM 7. *If ξ satisfies the equations* (15) *and is linearly dependent on $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, then $\xi = 0$.*

The proof is exactly that of Theorem 1. We shall define the Gramian of a set of complex quantities of finite norm precisely as was done in § 2.

THEOREM 8. *A necessary and sufficient condition that n complex quantities of finite norm be linearly dependent is that their Gramian vanish.*

The proof is precisely that of Theorem 2.

THEOREM 9. *If equations* (15) *are linearly independent, their general solution is given by formula* (5), *where η is any complex quantity such that the products $\alpha_1\bar{\eta}, \alpha_2\bar{\eta}, \cdots, \alpha_n\bar{\eta}$ all exist.*

The proof is practically identical with that of Theorem 3. In order that the solution ξ_1 have a finite norm it is necessary and sufficient, as we see from (3), that η have a finite norm.

Here, as in § 2, it is clear that two $\bar{\eta}$'s lead to the same solution ξ_1 when and only when their difference is linearly dependent upon $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$.

The requirement that η be so chosen that $\alpha_1\bar{\eta}, \alpha_2\bar{\eta}, \cdots, \alpha_n\bar{\eta}$ all exist will be fulfilled when η has a finite norm. It will, however, be fulfilled in many other cases. For example, denoting the components of α_i by a_{i1}, $a_{i2}, \cdots$, if all the a_{ij}'s are positive and a_{ij} constantly decreases and approaches zero with increasing j, we may take for η the complex quantity $(+1, -1, +1, -1, \cdots)$ whose norm is infinite.

Whenever ξ_1 has a finite norm, *i. e.*, whenever this is true of η, its norm is given by formula (7). As in §2 this formula may be now used to establish

THEOREM 10. *The Gramian of any number of linearly independent complex quantities of finite norm is real and positive.*

We now pass to the non-homogeneous equations:

$$\alpha_1\xi = b_1,\ \alpha_2\xi = b_2,\ \cdots,\ \alpha_n\xi = b_n, \tag{16}$$

the coefficients α_i again being assumed to have finite norms.

THEOREM 11. *If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are linearly independent, the equations* (16) *have one and only one solution linearly dependent upon $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$, and this solution is given by formula* (10).

The proof is precisely that of Theorem 5. The solution in question is termed the *principal solution.* The general solution of (16) is given by formula (11), where $\bar{\eta}$ is any complex quantity whose products with α_1, $\alpha_2, \cdots, \alpha_n$ exist.

THEOREM 12. *Among the solutions of* (16) *no other has so small a norm as the principal solution.*

The principal solution, being a linear combination of $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n$,

has a finite norm. This is also true of the general solution, $\xi = \xi_0 + \xi_1$, when and only when ξ_1 has a finite norm. From here on the proof is just like that of Theorem 6.

The norm of ξ_0 is given by formula (13).*

5. Some Theorems on the Limits of Complex Quantities. We proceed to establish some properties, which will be important for us, of complex quantities with an infinite number of components.†

If α and β have finite norms, we have from Theorems 8 and 10

$$G(\alpha, \bar{\beta}) = \begin{vmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{vmatrix} = |\alpha|^2 |\beta|^2 - |\alpha\bar{\beta}|^2 \geqq 0,$$

whence

(17) $$|\alpha\bar{\beta}| \leqq |\alpha| |\beta|.$$

Again, if $\gamma = \alpha + \beta$, we have, using (17) and remembering that $|\bar{\alpha}| = |\alpha|$,

$$|\gamma|^2 = (\alpha + \beta)(\bar{\alpha} + \bar{\beta}) = \alpha\bar{\alpha} + \alpha\bar{\beta} + \beta\bar{\alpha} + \beta\bar{\beta} \leqq |\alpha|^2 + 2|\alpha||\beta| + |\beta|^2,$$

(18) $$|\gamma| \leqq |\alpha| + |\beta|.$$

We next lay down the following

DEFINITIONS. *If $\alpha_n = (a_{n1}, a_{n2}, \cdots)$, $\alpha = (a_1, a_2, \cdots)$, we say that α_n converges to α as n becomes infinite when*

$$\lim_{n=\infty} a_{ni} = a_i \qquad (i = 1, 2, \cdots),$$

and write

$$\lim_{n=\infty} \alpha_n = \alpha.$$

We say that α_n has strong convergence toward α when, for all values of n greater than a certain number, $\alpha - \alpha_n$ has a finite norm, and

$$\lim_{n=\infty} |\alpha - \alpha_n| = 0,$$

and write, using Schmidt's notation,

$$\underline{\lim_{n=\infty}}\, \alpha_n = \alpha.$$

Strong convergence implies convergence. For if $\lim_{n=\infty} |\alpha - \alpha_n| = 0$, there exists, for every positive ϵ, an integer N such that

(19) $$\sum_{i=1}^{\infty} |a_i - a_{ni}|^2 < \epsilon^2, \qquad n > N,$$

* The theorem regarding bordered Gramians, stated in the footnote to formula (13), may now be generalized so as to apply to the Gramians of complex quantities with finite norms.

† Due to E. Schmidt, l. c., §§ 1–4. See also Kowalewski, l. c., § 165.

so that

(20) $$| a_i - a_{ni} | < \epsilon, \qquad \begin{Bmatrix} (i = 1, 2, \cdots), \\ n > N, \end{Bmatrix}$$

or

$$\lim_{n=\infty} a_{ni} = a_i \qquad (i = 1, 2, \cdots).*$$

If $\lim_{n=\infty} \alpha_n = \alpha$ and α_n has a finite norm when n is greater than a certain number, then α will have a finite norm. For (19) states that when $n > N$, $\alpha - \alpha_n$ has a finite norm; consequently the sum of α_n and $\alpha - \alpha_n$ has a finite norm.

Again, if $\lim_{n=\infty} \alpha_n = \alpha$, $\lim_{n=\infty} \beta_n = \beta$, then

(21) $$\lim_{n=\infty} (\alpha_n + \beta_n) = \alpha + \beta;$$

for we have seen that when $n > N$, $\alpha - \alpha_n$ and $\beta - \beta_n$ have finite norms, and hence from (18) we have

$$| \alpha + \beta - \alpha_n - \beta_n | \leqq | \alpha - \alpha_n | + | \beta - \beta_n |.$$

Furthermore if α_n, β_n have finite norms when $n > N$, so that α, β have finite norms,

(22) $$\lim_{n=\infty} \alpha_n \beta_n = \alpha\beta;$$

for when $n > N$, we have, using (17) and (18),

$$| \alpha\beta - \alpha_n\beta_n | = | (\alpha - \alpha_n)\beta + (\beta - \beta_n)\alpha - (\alpha - \alpha_n)(\beta - \beta_n) |$$

$$\leqq | \alpha - \alpha_n | | \beta | + | \beta - \beta_n | | \alpha | + | \alpha - \alpha_n | | \beta - \beta_n |.$$

Important special cases of (22) are

(23) $$\lim_{n=\infty} \alpha_n \beta = \alpha\beta;$$

(24) $$\lim_{n=\infty} \text{norm}\, \alpha_n = \text{norm}\, \alpha.$$

THEOREM 13. *A necessary and sufficient condition that $\lim_{n=\infty} \alpha_n$ exist is that, when n and m are any integers greater than a certain number, $\alpha_n - \alpha_m$ have a finite norm, and that to every positive ϵ there correspond an integer N such that*

(25) $$| \alpha_n - \alpha_m | < \epsilon, \qquad m, n > N.$$

The condition is necessary; for if $\lim_{n=\infty} \alpha_n = \alpha$, $| \alpha - \alpha_n | < \frac{1}{2}\epsilon$ when $n > N$. Hence when $m, n > N$

* We say that α_n converges *uniformly* toward α when for every positive ϵ there exists an N such that (20) is true. It is clear from the above that strong convergence implies uniform convergence, and uniform convergence implies convergence; but these implications do not hold in the reverse order.

$$|\alpha_n - \alpha_m| = |\alpha_n - \alpha + \alpha - \alpha_m| \leqq |\alpha_n - \alpha| + |\alpha - \alpha_m| < \epsilon.$$

To show the sufficiency of the condition we first observe that if (25) holds,

$$\sum_{k=1}^{\infty} |a_{nk} - a_{mk}|^2 < \epsilon^2, \qquad m, n > N,$$

and hence

$$|a_{nk} - a_{mk}| < \epsilon, \qquad \begin{Bmatrix} (k = 1, 2, \cdots), \\ m, n > N. \end{Bmatrix}$$

This shows that $\lim_{n=\infty} a_{nk}$ exists: denote it by a_k. Then as

$$\sum_{k=1}^{p} |a_{nk} - a_{mk}|^2 < \epsilon^2, \qquad m, n > N,$$

$$\lim_{n=\infty} \sum_{k=1}^{p} |a_{nk} - a_{mk}|^2 = \sum_{k=1}^{p} |a_k - a_{mk}|^2 \leqq \epsilon^2, \qquad m > N.$$

As this holds for every p, we have

$$\sum_{k=1}^{\infty} |a_k - a_{mk}|^2 \leqq \epsilon^2, \qquad m > N;$$

or, upon writing $\alpha = (a_1, a_2, \cdots)$,

$$\lim_{m=\infty} |\alpha - \alpha_m| = 0$$

as we wished to prove.

COROLLARY. *When condition (25) is fulfilled and α_n is always of finite norm, α is also of finite norm.*

DEFINITION. *The infinite series of complex quantities $\alpha_1 + \alpha_2 + \cdots$ is said to converge strongly to a complex quantity σ when σ_n converges strongly to σ, where $\sigma_n = \alpha_1 + \cdots + \alpha_n$.*

From Theorem 13 we see that a necessary and sufficient condition for the strong convergence of the above series is that after a certain point the terms of the series all have finite norms and that, to every positive ϵ, there correspond an integer N such that

(26) $$|\sigma_n - \sigma_m| = |\alpha_{m+1} + \alpha_{m+2} + \cdots + \alpha_n| < \epsilon, \qquad m, n > N.$$

DEFINITION. *Two complex quantities α, β, are said to be orthogonal if $\alpha\bar{\beta}$, and hence also $\bar{\alpha}\beta$, is zero.*

If the α's have finite norms and are mutually orthogonal, we may, by squaring (26), readily reduce it to the form

$$|\alpha_{m+1}|^2 + |\alpha_{m+2}|^2 + \cdots + |\alpha_n|^2 < \epsilon^2, \qquad m, n > N.$$

This being precisely a necessary and sufficient condition that the series $|\alpha_1|^2 + |\alpha_2|^2 + \cdots$ converge, we have proved

THEOREM 14. *A series of mutually orthogonal complex quantities of*

*finite norm is strongly convergent when and only when the series of their norms converges.**

Furthermore as $|\sigma_n|^2 = |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2$, we infer from (24) the

COROLLARY. *If the conditions of Theorem 14 are fulfilled, the norm of the series is equal to the series of the norms of the terms.*

6. System of an Infinite Number of Linear Equations in an Infinite Number of Variables. We are now in position to consider the infinite system of homogeneous equations in an infinite number of variables

$$\alpha_1\xi = 0, \quad \alpha_2\xi = 0, \quad \cdots, \tag{27}$$

where

$$\alpha_i = (a_{i1}, a_{i2}, \cdots) \qquad (i = 1, 2, \cdots),$$

$$\xi = (x_1, x_2, \cdots).$$

We assume that all the coefficients α_i have finite norms and none of them are linearly dependent. The general solution, $\xi_1^{(n)}$, of the first n of these equations is given by formula (5)

$$\xi_1^{(n)} = \sum_{i=1}^{n} c_i^{(n)}\bar\alpha_i + \bar\eta = \frac{\begin{vmatrix} \alpha_1\bar\alpha_1 & \alpha_1\bar\alpha_2 & \cdots & \alpha_1\bar\alpha_n & \alpha_1\bar\eta \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar\alpha_1 & \alpha_n\bar\alpha_2 & \cdots & \alpha_n\bar\alpha_n & \alpha_n\bar\eta \\ \bar\alpha_1 & \bar\alpha_2 & \cdots & \bar\alpha_n & \bar\eta \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{28}$$

Here $(c_1^{(n)}, c_2^{(n)}, \cdots, c_n^{(n)})$ is a solution of equations (4).

We wish to show that $\xi_1^{(n)}$ converges strongly to a limit as $n = \infty$; and to this end we proceed to throw it into the form

$$\xi_1^{(n)} = \xi_1^{(1)} + (\xi_1^{(2)} - \xi_1^{(1)}) + \cdots + (\xi_1^{(n)} - \xi_1^{(n-1)}).$$

If we write

$$c_i^{(n)} - c_i^{(n-1)} = z_i^{(n)} \qquad (i = 1, 2, \cdots, n-1),$$

$$c_n^{(n)} = z_n^{(n)},$$

and subtract from the first $n - 1$ equations (4) the similar equations satisfied by $(c_1^{(n-1)}, \cdots, c_{n-1}^{(n-1)})$, we find that the z's satisfy the $n - 1$ *homogeneous* equations

$$\begin{cases} \alpha_1\bar\alpha_1 z_1^{(n)} + \alpha_1\bar\alpha_2 z_2^{(n)} + \cdots + \alpha_1\bar\alpha_n z_n^{(n)} = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \alpha_{n-1}\bar\alpha_1 z_1^{(n)} + \alpha_{n-1}\bar\alpha_2 z_2^{(n)} + \cdots + \alpha_{n-1}\bar\alpha_n z_n^{(n)} = 0. \end{cases}$$

* The proof above establishes the more general theorem in which the condition of orthogonality is replaced by the condition $\alpha_i\bar\alpha_j + \alpha_j\bar\alpha_i = 0$ or (real part of $\alpha_i\,\bar\alpha_j) = 0$ when $i \neq j, i, j = 1, 2, \cdots$.

Moreover we have

$$\xi_1^{(n)} - \xi_1^{(n-1)} = z_1^{(n)}\bar{\alpha}_1 + z_2^{(n)}\bar{\alpha}_2 + \cdots + z_n^{(n)}\bar{\alpha}_n \qquad (n = 2, 3, \cdots).$$

Solving the homogeneous equations for the z's and substituting in the last equation, we have

(29) $$\xi_1^{(n)} - \xi_1^{(n-1)} = k_n\varphi_n,$$

where k_n is an undetermined scalar and

$$\varphi_n \equiv \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\alpha_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{n-1}\bar{\alpha}_1 & \alpha_{n-1}\bar{\alpha}_2 & \cdots & \alpha_{n-1}\bar{\alpha}_n \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n \end{vmatrix} \qquad (n = 2, 3, \cdots).$$

Multiplying both sides of (29) by α_n and using (28), we find $-H_n/G_{n-1} = k_nG_n$, where, for brevity, we have written

$$H_n = \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_{n-1} & \alpha_1\bar{\eta} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{n-1}\bar{\alpha}_1 & \alpha_{n-1}\bar{\alpha}_2 & \cdots & \alpha_{n-1}\bar{\alpha}_{n-1} & \alpha_{n-1}\bar{\eta} \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_{n-1} & \alpha_n\bar{\eta} \end{vmatrix} \qquad (n = 2, 3, \cdots),$$

$$G_n = G(\alpha_1, \cdots, \alpha_n).$$

Therefore

$$\xi_1^{(n)} - \xi_1^{(n-1)} = -\frac{H_n}{G_{n-1}G_n}\varphi_n;$$

and as $\xi_1^{(1)} = \bar{\eta} - (\alpha_1\bar{\eta}/\alpha_1\bar{\alpha}_1)\bar{\alpha}_1$, we have, if we set $\varphi_1 = \bar{\alpha}_1$, $G_0 = 1$, $H_1 = \alpha_1\bar{\eta}$,

(30) $$\xi_1^{(m)} = \bar{\eta} - \sum_{n=1}^{n=m}\frac{H_n}{G_{n-1}G_n}\varphi_n.$$

If η, and hence $\xi_1^{(m)}$, has a finite norm, we see from (6) that norm $\xi_1^{(m)} = \eta\xi_1^{(m)}$. Assuming, then, that this is the case, we have, since $\varphi_n\eta = \bar{H}_n$,

(31) $$\text{norm } \xi_1^{(m)} = |\eta|^2 - \sum_{n=1}^{n=m}\frac{|H_n|^2}{G_{n-1}G_n}.$$

The series of positive or zero terms

(32) $$\sum_{n=1}^{\infty}\frac{|H_n|^2}{G_{n-1}G_n}$$

is therefore convergent for every η of finite norm since the sum of its first m terms is by (31) not greater than $|\eta|^2$.

We next note that the terms of the series of complex quantities of finite norm

$$(33) \qquad \sum_{n=1}^{\infty} \frac{H_n}{G_{n-1}G_n} \varphi_n$$

are mutually orthogonal; for as

$$\varphi_n \alpha_i = 0 \qquad (i = 1, 2, \cdots, n-1),$$

we have

$$\varphi_n \bar{\varphi}_m = 0, \qquad m < n.$$

By Theorem (14) the series (33) will converge strongly if the series of the norms of its terms converges. If we use the relation

$$\varphi_n \bar{\varphi}_n = \varphi_n \alpha_n G_{n-1} = G_n G_{n-1},$$

this series of norms proves to be precisely (32), which we have just shown to be convergent when η is of finite norm. Hence series (33) converges strongly when η has a finite norm, as does likewise the series

$$(34) \qquad \xi_1 \equiv \lim_{m=\infty} \xi_1^{(m)} = \bar{\eta} - \sum_{n=1}^{\infty} \frac{H_n}{G_{n-1}G_n} \varphi_n = \bar{\eta} - \sum_{n=1}^{\infty} \frac{\bar{\eta}\bar{\varphi}_n}{\phi_n \bar{\varphi}_n} \varphi_n.$$

ξ_1 is a solution of equations (27) having a finite norm. For consider any one of these equations, say $\alpha_k \xi = 0$; since

$$\alpha_k \xi_1^{(m)} = 0 \qquad (m = k, k+1, \cdots),$$

we have from (23)

$$\lim_{m=\infty} (\alpha_k \xi_1^{(m)}) = \alpha_k \xi_1 = 0.$$

That ξ_1 is of finite norm follows from the fact that $\xi_1^{(m)}$ is always of finite norm and converges *strongly* towards ξ_1.

Conversely, if ξ_1 is any solution of equations (27), we may obtain it by letting $\bar{\eta} = \xi_1$ in the formula (34), for then all the terms after the first vanish. Thus we have proved

THEOREM 15. *If η is a complex quantity of finite norm, $\xi_1^{(n)}$, given by formula* (28), *approaches a limiting complex quantity of finite norm as n becomes infinite, and this limit, ξ_1, is a solution of the equations* (27).

Conversely, every solution of (27), *whether of finite norm or not, can be obtained by properly choosing η in* (34).

From formulas (24) and (31) we have

$$(35) \qquad \text{norm } \xi_1 = \lim_{m=\infty} \text{norm } \xi_1^{(m)} = |\eta|^2 - \sum_{m=1}^{\infty} \frac{|H_n|^2}{G_{n-1}G_n}$$

whenever η is of finite norm. Referring to (7), we see that this may also be written as

$$(36) \qquad \text{norm } \xi_1 = \lim_{n=\infty} \frac{G(\alpha_1, \alpha_2, \cdots, \alpha_n, \eta)}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}.$$

We turn now to the non-homogeneous equations

$$\alpha_1\xi = b_1, \quad \alpha_2\xi = b_2, \quad \cdots, \tag{37}$$

where we again assume that all the coefficients α_i have finite norms and none of them are linearly dependent. The principal solution of the first n of these equations, which we will denote by $\xi_0^{(n)}$, is given by formula (10)

$$\xi_0^{(n)} = \sum_{i=1}^{n} c_i^{(n)}\bar{\alpha}_i = \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & -b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & -b_n \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n & 0 \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}. \tag{38}$$

Here $(c_1^{(n)}, c_2^{(n)}, \cdots, c_n^{(n)})$ is a solution of the equations obtained from (4) by replacing their right-hand members, $-\alpha_1\bar{\eta}$, $-\alpha_2\bar{\eta}$, $\cdots$, $-\alpha_n\bar{\eta}$ by $b_1, b_2, \cdots, b_n$ respectively. A consideration of the process by which $\xi_1^{(n)} - \xi_1^{(n-1)}$ was obtained shows that we may obtain $\xi_0^{(n)} - \xi_0^{(n-1)}$ from this expression by replacing $-\alpha_1\bar{\eta}$, $-\alpha_2\bar{\eta}$, $\cdots$, $-\alpha_n\bar{\eta}$ by $b_1, b_2, \cdots, b_n$ respectively; consequently in place of $-H_n$ we must now introduce the determinant

$$B_1 = b_1, \quad B_n \equiv \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_{n-1} & b_1 \\ \alpha_2\bar{\alpha}_1 & \alpha_2\bar{\alpha}_2 & \cdots & \alpha_2\bar{\alpha}_{n-1} & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_{n-1} & b_n \end{vmatrix} \quad (n = 2, 3, \cdots)$$

and we obtain

$$\xi_0^{(n)} - \xi_0^{(n-1)} = \frac{B_n}{G_{n-1}G_n}\varphi_n.$$

As

$$\xi_0^{(1)} = \frac{b_1}{\alpha_1\bar{\alpha}_1}\bar{\alpha}_1,$$

$$\xi_0^{(m)} = \sum_{n=1}^{n=m} \frac{B_n}{G_{n-1}G_n}\varphi_n. \tag{39}$$

We are thus led to consider the series

$$\sum_{n=1}^{\infty} \frac{B_n}{G_{n-1}G_n}\varphi_n \tag{40}$$

whose terms are mutually orthogonal complex quantities of finite norm—as we know from the previously established properties of φ_n. By Theorem 14 this series will be strongly convergent when and only when the series of the norms of its terms

$$(41) \qquad \sum_{n=1}^{\infty} \frac{|B_n|^2}{G_{n-1}G_n}$$

converges. Thus when series (41) converges we have

$$(42) \qquad \xi_0 \equiv \lim_{m=\infty} \xi_0^{(m)} = \sum_{n=1}^{\infty} \frac{B_n}{G_{n-1}G_n} \varphi_n$$

and an argument similar to that which follows (34) shows that ξ_0 is a solution of equations (37) having a finite norm. Now if equations (37) have *any* solution, ξ, of finite norm, then, as $\xi_0^{(m)}$ is the solution of least norm of the first m of these equations,

$$\text{norm } \xi_0^{(m)} \leqq \text{norm } \xi;$$

and since norm $\xi_0^{(m)}$ proves to be precisely the sum of the first m terms of (41), the convergence of this series is established. Thus we have proved

THEOREM 16. *A necessary and sufficient condition that equations* (37) *have a solution of finite norm is that the series* (41) *converge. When this is the case,* $\xi_0^{(n)}$, *given by formula* (38), *approaches strongly a limiting complex quantity of finite norm as n becomes infinite, and this limit,* ξ_0, *is a solution of the equations.*

ξ_0 is termed the *principal solution* of (37). We may form the general solution by adding to the particular solution ξ_0 the general solution ξ_1 of equations (27):

$$(43) \qquad \xi = \xi_0 + \xi_1 = \lim_{n=\infty} \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & \alpha_1\bar{\eta} - b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & \alpha_n\bar{\eta} - b_n \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_n & \bar{\eta} \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}.$$

From the Corollary to Theorem 14 we have

$$\text{norm } \xi_0 = \sum_{n=1}^{\infty} \frac{|B_n|^2}{G_{n-1}G_n},$$

or, referring to (13),

$$(44) \qquad \text{norm } \xi_0 = \lim_{n=\infty} - \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_n & b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \alpha_n\bar{\alpha}_2 & \cdots & \alpha_n\bar{\alpha}_n & b_n \\ \bar{b}_1 & \bar{b}_2 & \cdots & \bar{b}_n & 0 \end{vmatrix}}{G(\alpha_1, \alpha_2, \cdots, \alpha_n)}.$$

If ξ_1 has a finite norm, the same is true of ξ, and

$$\text{norm } \xi = \text{norm } \xi_0 + \text{norm } \xi_1,$$

for from (38) $\xi_0^{(n)}\bar{\xi}_1 = 0$, so that upon applying (23), $\xi_0\bar{\xi}_1 = \xi_1\bar{\xi}_0 = 0$. Consequently

$$\text{norm } \xi \geqq \text{norm } \xi_0$$

the sign of equality holding only when $\xi_1 = 0$, in which case $\xi = \xi_0$. Thus we have proved

THEOREM 17. *Among the solutions of* (37) *no other has so small a norm as the principal solution.*

7. Some further facts.—The general solution ξ_1 of the homogeneous equations (27) is a function of the complex parameter η

$$\xi_1 = \psi(\eta).$$

A glance at (28) shows us at once that ψ is, in an extended sense, a *linear* function; that is

THEOREM 18. *If $\eta', \eta'', \cdots, \eta^{[k]}$ are complex quantities with finite norms and $c_1, \cdots, c_k$ are scalars, then*

$$\psi(c_1\eta' + \cdots + c_k\eta^{[k]}) = c_1\psi(\eta') + \cdots + c_k\psi(\eta^{[k]}).$$

A further important fact is that ψ has *strong continuity* for every value of η with finite norm; that is

THEOREM 19. *If η' has a finite norm, then as η approaches η' strongly, $\psi(\eta)$ approaches $\psi(\eta')$ strongly.*

To prove this, we derive from Theorem 18 and from (35) the relation

$$\text{norm } [\psi(\eta') - \psi(\eta)] = \text{norm } \psi(\eta' - \eta) \leqq \text{norm } (\eta' - \eta),$$

from which our theorem follows at once.

Let us now denote the components of η by $y_1, y_2, \cdots$, and the complex quantity whose first n components are $y_1, \cdots, y_n$ while all its subsequent components are zero by η_n. Then, if η is of finite norm,

$$(45) \qquad \lim_{n=\infty} \eta_n = \eta.$$

For norm $(\eta - \eta_n) = |y_{n+1}|^2 + |y_{n+2}|^2 + \cdots$, and, this being the remainder of a convergent series, approaches zero as n becomes infinite.

Let us denote by ϵ_i the complex quantity whose ith component is 1 while all its other components are zero. Then

$$\psi(\epsilon_i) = \lim_{n=\infty} \frac{\begin{vmatrix} \alpha_1\bar{\alpha}_1 & \cdots & \alpha_1\bar{\alpha}_n & a_{1i} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_n\bar{\alpha}_1 & \cdots & \alpha_n\bar{\alpha}_n & a_{ni} \\ \bar{\alpha}_1 & \cdots & \bar{\alpha}_n & \epsilon_i \end{vmatrix}}{G(\alpha_1, \cdots, \alpha_n)}.$$

THEOREM 20. *A necessary and sufficient condition that the homogeneous system* (27) *have no solution of finite norm except zero is that all the quantities* $\psi(\epsilon_i)$ *be zero.*

That this is a necessary condition is obvious. To prove it sufficient assume $\psi(\epsilon_i) = 0$ $(i = 1, 2, \cdots)$. By Theorem 18, $\psi(\eta) = 0$ whenever η has only a finite number of components different from zero. But, by (45), every η of finite norm is the strong limit of such a set of η's. Consequently, by Theorem 19, $\psi(\eta) = 0$ for every η of finite norm, as was to be proved.

We have expressed the solutions ξ_1 and ξ_0 as well as their norms, as the limit of the ratio of two determinants of order $n + 1$ and n as n becomes infinite. We proceed to inquire under what conditions the individual determinants, and not merely their ratios, converge. In all cases the denominator determinant is $G(\alpha_1, \cdots, \alpha_n)$, and if this Gramian converges as n becomes infinite, the determinants in the numerators will likewise converge. Thus we have merely to consider the convergence of $G(\alpha_1, \cdots, \alpha_n)$ as n becomes infinite, or, as we phrase it, the convergence of the *infinite Gramian*, $G(\alpha_1, \alpha_2, \cdots)$.

THEOREM 21. *A sufficient condition for the convergence of the infinite Gramian of the complex quantities* $\alpha_1, \alpha_2, \cdots$ *which have finite norms is that the infinite product* $\prod_{i=1}^{\infty} |\alpha_i|^2$ *diverge to zero or converge.*

Consider the set of complex quantities $\beta_i = \alpha_i/|\alpha_i|$ whose norms are all unity. We have, then,

$$G(\alpha_1, \cdots, \alpha_n) = G(\beta_1, \cdots, \beta_n) \prod_{i=1}^{n} |\alpha_i|^2. \tag{46}$$

Now

$$G(\beta_1, \cdots, \beta_n) = \begin{vmatrix} \beta_1\bar{\beta}_1 & \cdots & \beta_1\bar{\beta}_{n-1} & \beta_1\bar{\beta}_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \beta_{n-1}\bar{\beta}_1 & \cdots & \beta_{n-1}\bar{\beta}_{n-1} & \beta_{n-1}\bar{\beta}_n \\ \beta_n\bar{\beta}_1 & \cdots & \beta_n\bar{\beta}_{n-1} & 0 \end{vmatrix} + |\beta_n|^2 G(\beta_1, \cdots, \beta_{n-1}).$$

The first term on the right is a bordered Gramian of the form of the numerator of (13) and is therefore negative or zero (see footnote at the end of § 3). Consequently

$$G(\beta_1, \cdots, \beta_n) \leqq G(\beta_1, \cdots, \beta_{n-1});$$

and since $G(\beta_1, \cdots, \beta_n)$ is never negative, $\lim_{n=\infty} G(\beta_1, \cdots, \beta_n)$ exists. Thus when $\prod_{i=1}^{\infty} |\alpha_i|^2$ diverges to zero or converges, we have from (46) that $G(\alpha_1, \alpha_2, \cdots)$ converges, as we wished to prove.

COROLLARY 1. *If* $G(\beta_1, \beta_2, \cdots) \neq 0$ *the condition that* $\prod_{i=1}^{\infty} |\alpha_i|^2$ *diverge to zero or converge is also necessary for the convergence of* $G(\alpha_1, \alpha_2, \cdots)$.

COROLLARY 2. *If* $\prod_{i=1}^{\infty} |\alpha_i|^2 = 0$, *then* $G(\alpha_1, \alpha_2, \cdots) = 0$.

We also note that $G(\alpha_1, \alpha_2, \cdots) = 0$ when any of the complex quantities α_i are linearly dependent.

From Theorem 21 we now see that the determinants occurring in the expressions for $\xi_1^{(n)}$ and $\xi_0^{(n)}$ (and for their norms) will converge as $n = \infty$ if, at the start, the equations (27) and (37) respectively are divided through by scalars so as to make the norms of all the α's $\leqq 1$. If, when this is done, $G(\alpha_1, \alpha_2, \cdots) \neq 0$, the formulæ for ξ_1 and ξ_0 furnish solutions for these infinite systems of equations in terms of infinite determinants, properly so called. Of course the last row and column of the numerator determinants must then be written as first row and column.

CAMBRIDGE, MASS. AND CINCINNATI, OHIO,
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ON THE THEORY OF CORRELATION WITH SPECIAL REFERENCE TO CERTAIN SIGNIFICANT LOCI ON THE PLANE OF DISTRIBUTION IN THE CASE OF NORMAL CORRELATION.

By H. L. Rietz.

1. Introduction.—The notion of correlation is of such importance in science that it seems it should become almost as familiar to the scientist as the notions of a mathematical function and of independence in the probability sense. The purposes of the present paper are (1) to present the elements of a theory of correlation from assumptions that are suggested by applications and that seem to appeal to the mathematical student beginning the study of statistics, (2) to give a few properties of normally correlated statistical data by means of curves or contour lines on what I call the plane of distribution.

Take X and Y to represent associated classes of individuals that have definite values or with attributes that have definite values. These classes may represent any one of a great variety of concrete situations. To illustrate, X may represent rainfalls at a given place in months of April and Y those in months of June; X and Y may refer to fathers and sons, when the inheritance of some character is in question; X and Y may represent statures of husbands and of their wives; X may represent hours per day worked by laborer, and Y the corresponding wages paid. These illustrations very naturally suggest the following questions that show the purpose of a theory of correlation: Is there a measurable tendency for wet Aprils to be followed by dry Junes? To what extent do a class of men, in general, resemble their father with respect to some character, say stature? Do tall men, in general, marry tall wives? Do high wages go with short hours of labor?

The problem that we set is to describe, by some summary method, the tendency of corresponding individuals of X and Y to vary in the same or in opposite directions, when the variations are to be attributed to an indefinite number of unassignable causes. It is well known that the relationship of corresponding individuals of X and Y, in the illustrations cited above, is not such a perfect dependence as is given by a mathematical function. When a value is assigned to X, the corresponding values of Y have a certain amount of freedom. They may, however, be far from free in the probability sense of freedom or independence.

A relation expressed by a mathematical function and independence in the probability sense may be regarded as two extremes between which there exists a large region for a theory of correlation.

2. Table of Double Classification.—To be precise, let

$$x_1,\ x_2,\ x_3,\ \cdots,\ x_n$$

be values of the individuals of a random sample of n drawn from the class X. In the language of statistics, we call these x's variates. Let y's represent the corresponding variates of the class Y. That is to say, X and Y are associated so that

$$(x_1,\ y_1),\ (x_2,\ y_2),\ (x_3,\ y_3),\ \cdots,\ (x_n,\ y_n)$$

are pairs of variates in correspondence.

The first step in the description of the dependence of the two attributes is the construction of a table* of double classification of the following form:

	X_1	X_2	X_3	......	......	......	X_s	......	......	......	......	X_i	
Y_m	N_{1m}	N_{2m}	N_{3m}	......	......	......	N_{sm}	......	......	......	......	N_{im}	N_{ym}
$\vdots$	......	......	......	......	......	......	......	......	......	......	......	......	......
Y_t	N_{1t}	N_{2t}	N_{3t}	......	......	......	N_{st}	......	......	......	......	N_{it}	N_{yt}
$\vdots$	......	......	......	......	......	......	......	......	......	......	......	......	......
$\vdots$	......	......	......	......	......	......	......	......	......	......	......	......	......
Y_3	N_{13}	N_{23}	N_{33}	......	......	......	N_{s3}	......	......	......	......	N_{i3}	N_{y3}
Y_2	N_{12}	N_{22}	N_{32}	......	......	......	N_{s2}	......	......	......	......	N_{i2}	N_{y2}
Y_1	N_{11}	N_{21}	N_{31}	......	......	......	N_{s1}	......	......	......	......	N_{i1}	N_{y1}
	N_{x1}	N_{x2}	N_{x3}	......	......	......	N_{xs}	......	......	......	......	N_{xi}	n

FIG. 1.

The symbols X_1, X_2, $\cdots$ in the top row of the table mark subclasses covering, in general, equal intervals on the range that includes the entire class X. Similarly, Y_1, Y_2, in the column on the left mark subclasses that are taken to include, in general, equal intervals of the range that includes the entire class Y. The number of such subclasses may be two or more.

In this table (Fig. 1), any number, say N_{st} $(s \neq x,\ s \neq y)$, indicates the number that belongs to both subclasses X_s and Y_t. The vertical column of frequencies $N_{s1}, N_{s2}, N_{s3}, \cdots, N_{st}, \cdots$ corresponding to any mark X_s is called an X-array of y's. Similarly, any row of frequencies N_{1t}, N_{2t}, N_{3t}, $\cdots$, N_{it}, $\cdots$ corresponding to any mark Y_t is a y-array of x's.

The numbers in the lower row give the sums of numbers N in columns, and show the frequency distribution of the total sample of n into subclasses X_1, X_2, $\cdots$. Likewise, the column of totals on the right is the frequency distribution into subclasses Y_1, Y_2, $\cdots$.

* Such a table used to study the correlation of statistical data seems to have been employed first by Francis Galton, Proc. Royal Society, XL, p. 68.

3. Test of Independence.—If the two classes X and Y are independent in the probability sense, the best estimate (from our sample of n) that can be given, from the separate classes, of the probability that a pair (x_p, y_p) taken at random is such that x_p belongs to the subclass X_s and y_p to Y_t, is the product

$$\frac{N_{xs} \cdot N_{yt}}{n^2}. \tag{1}$$

But, from the table (Fig. 1), the best estimate that a pair belongs to both these classes is N_{st}/n. If the deviations

$$\delta = \frac{N_{st}}{n} - \frac{N_{xs} \cdot N_{yt}}{n^2}, \tag{2}$$

when extended to all compartments of the table, are greater than may be attributed to fluctuations in drawing random samples, there is lack of independence. If the differences (2) are to be attributed to smallness of the sample, we say the difference is insignificant. A test as to whether the deviations δ are to be regarded as significant has been given by Sheppard.*

If the classes X and Y are not independent, and are not absolutely dependent in the sense that a mathematical function determines the one when the other is given; then, we may say, in a general way, that a theory of correlation is required to describe the association of the two classes X and Y.

4. General Description.—In many applications, where the correlation is considerable, a certain amount of progress in treating the association of X and Y can be made by a consideration of the arrays of the double entry table of values without the use of special mathematical methods. That is to say, we may treat each array as a frequency distribution, and find some kind of average values for the arrays and for the variability from this average. Such a treatment is, however, obviously inadequate for many purposes.

Suppose we erect at the center of each rectangle (Fig. 1) of the table a perpendicular to the plane of the rectangle proportional in length to the number in that rectangle. The ends of these perpendiculars suggest a surface to describe the distribution.

5. Geometrical Description.—To follow the suggestion just mentioned, it is convenient to use (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n) as deviations from the means of variates of the respective classes X and Y rather than for the actual values of the variates themselves. They will be used in this sense throughout the remainder of the paper.

* Philosophical Transactions of the Royal Society, vol. 192A, 1899, p. 128.

Represent these deviations (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n) from the mean values as coördinates of points in a plane (Fig. 2*). I call this plane the *plane of distribution*. It is the purpose of a theory of correlation to characterize the arrangement of such points without special reference to individual points. Divide the range along the x-axis that would include all points of the class into equal intervals Δx. Likewise, divide the range along the Y-axis into equal intervals Δy. The points included by two parallels to the Y-axis (AB) may be said to constitute an x-array of Y's. Similarly, the points included by parallels $(A'B')$ are said to be a Y-array of x's.

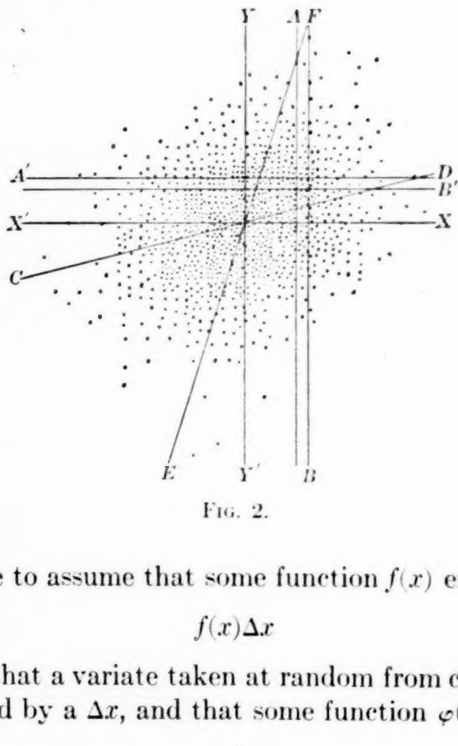

FIG. 2.

It is reasonable to assume that some function $f(x)$ exists such that

$$f(x)\Delta x \tag{3}$$

is the probability that a variate taken at random from class X gives a point in an array marked by a Δx, and that some function $\varphi(y)$ is such that

$$\varphi(y)\Delta y \tag{4}$$

is the probability that a variate taken at random from Y gives a point in an array marked Δy. The table of double classification (Fig. 1) with perpendiculars to the plane at the centers of rectangles suggested the idea of a surface. To follow this suggestion, we consider an area $\Delta x \Delta y$ at the intersection of any two arrays marked Δx and Δy. It may be assumed that

* This distribution of points is not made up in an entirely artificial manner, but represents approximately the distribution of a class of husbands and wives with respect to stature. See Pearson and Lee, Biometrika, vol. 2, p. 408.

a function $z = \psi(x, y)$ exists such that

$$\psi(x, y)\Delta x\Delta y \tag{5}$$

gives the probability that a pair (x_p, y_p) taken at random gives a point in $\Delta x\Delta y$.

We may obviously add the conditions

$$\int_{-\infty}^{+\infty} f(x)dx = 1, \quad \int_{-\infty}^{+\infty} \varphi(y)dy = 1, \quad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi(x, y)dxdy = 1, \tag{6}$$

if the functions in (3), (4), and (5) are integrable from $-\infty$ to $+\infty$.

To render more precise and useful the idea of independence discussed in § 3, we may now use the notation of § 5, and define that the two classes X and Y are independent if

$$\psi(x, y) = f(x)\varphi(y) \tag{7}$$

is an identity. If this equality is not an identity, the two classes are correlated. From this general negative definition of correlaton, it seems difficult to determine the character of $\psi(x, y)$ so as to give it a practical value. We seek therefore an affirmative definition at some loss of generality. To obtain a valuable affirmative definition, we fix our attention on the mean values of those variates of one class, say of Y, that are in an array marked Δx. Or, we may regard the points (Fig. 2) as particles of equal mass and fix our attention on the centroid of these particles. With the limiting case of an indefinitely large number of variates and small values of Δx, it seems reasonable, since mean values have variations of a higher order of smallness than individuals, that these centroids, in general, arrange themselves along a smooth curve. That is to say, there may be a correspondence, given by a mathematical function, between any assigned X and the centroid of corresponding points of Y.

With reference to the surface

$$z = \psi(x, y),$$

this means that the y coördinate, $\bar{y}$, of the centroid of any section by assigning x is a function of x. Suppose that

$$\bar{y} = \frac{\int_{-\infty}^{+\infty} y\psi(x, y)dy}{\int_{-\infty}^{+\infty} \psi(x, y)dy} = \theta(x) \tag{8}$$

for any assigned x.

As an affirmative, though special definition of correlation, we define that the class Y is correlated with X if $\theta(x)$ is different from zero. As we have

selected the origin so that $\theta(x)$ cannot be a constant different from zero, the simplest case of correlation we have to consider is that for which

$$\theta(x) = mx + b, \tag{9}$$

a linear function of x.

It affords a vivid description to interpret $\theta(x)$ on the plane of distribution (Fig. 2). It is sometimes called the curve of regression of y's on x's. Moreover, when $\theta(x)$ is linear, it is said that there is linear regression. Obviously, another curve of regression of x's on y's exists.

To obtain the m and b to apply to any numerical case, we make use of our sample of n. If the line (9) is subjected to the least squares condition that m and b are to be determined so that the sum of the squares of its deviations (measured parallel to the y-axis) from the means of arrays (weighted with the number in the arrays) is to be a minimum, we obtain

$$y = r\frac{\sigma_y}{\sigma_x}x,* \tag{10}$$

where σ_x^2 is the mean square of $x_1, x_2, \cdots, x_n$,
σ_y^2 is the mean square of $y_1, y_2, \cdots, y_n$,
and r is defined by the formula

$$r = \frac{\sum_{q=1}^{q=n} x_q y_q}{n\sigma_x\sigma_y}.$$

The σ's are called the "standard deviations" of the systems of variates and r is called the "correlation coefficient."

The line (10) is the line of distribution of the means of y's that correspond to assigned x's. By analogy,

$$x = r\frac{\sigma_x}{\sigma_y}y \tag{11}$$

is the line of regression of x's on y's. From our special definition, there is no correlation if $r = 0$.

7. Standard Deviation of Arrays.—The mean square of the deviations of all points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, [Fig. 2], from corresponding points

$$\left(x_1, r\frac{\sigma_y}{\sigma_x}x_1\right), \quad \left(x_2, r\frac{\sigma_y}{\sigma_x}x_2\right), \quad \cdots, \quad \left(x_n, r\frac{\sigma_y}{\sigma_x}x_n\right)$$

on the line

$$y = r\frac{\sigma_y}{\sigma_x}x$$

is given by

* See Fig. 2. Line CD is the line of regression of wives with respect to husbands, and EF is the line of regression of husbands with respect to wives.

$$\frac{\sum_{q=1}^{q=n}\left(y_q - r\frac{\sigma_y}{\sigma_x}x_q\right)^2}{n} = \sum_{q=1}^{q=n}\frac{y_q^2}{n} - \frac{2\sigma_y r}{\sigma_x}\frac{\sum_{q=1}^{q=n}x_q y_q}{n} + \frac{r^2\sigma_y^2}{\sigma_x^2}\frac{\sum_{q=1}^{q=n}x_q^2}{n} \tag{12}$$
$$= \sigma_y^2 - 2r^2\sigma_y^2 + r^2\sigma_y^2$$
$$= \sigma_y^2(1 - r^2).$$

Suppose that the regression is linear, so that the centroid of the x-arrays of y's may be taken on the line

$$y = r\frac{\sigma_y}{\sigma_x}x,$$

and that the standard deviations of x-arrays of y's are equal. Then $\sigma_y^2(1 - r^2)$ becomes the square of the standard deviation of each x-array of y's. When the standard deviations of these arrays are unequal, $\sigma_y^2(1 - r^2)$ is merely a sort of an average value of the square of standard deviations of arrays.

Similarly, $\sigma_x^2(1 - r^2)$ is the square of the standard deviation of a y-array of x's.

8. The Normal Correlation Surface.—The normal correlation surface dates back to a memoir by Bravais* in 1846. The importance of this surface in the mathematics of statistics was first recognized by Galton† and has been fully demonstrated by the work of Pearson,‡ Edgeworth, and Yule.

The equation of the normal surface may be written in the form

$$z = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - r^2}}e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2rxy}{\sigma_x\sigma_y}\right)}. \tag{13}$$

For certain associated classes X and Y, this is the form of $\psi(x, y)$[§ 5, (5)]. When $\psi(x, y)$ takes this form, the correlation is said to be normal.

It is easy to give many properties of this surface from certain sets of which equation (13) may be derived. One set of conditions that characterize the surface and that suggest themselves very naturally in attempting a description of certain double entry tables of classification may be stated as follows:

(a) The regression of at least one set of variates on the other is linear. Applied to the surface, this condition means that

$$\theta(x) = r\frac{\sigma_y}{\sigma_x}x. \qquad [\S\, 6, (8)]$$

* Sur les Probabilitès des Erreurs de Situatión d'un Point. Memoires par divers Savants a l'Académie des Sciences de France, t. IX (1846), pp. 255–332.

† Galton, Proc. Roy. Soc., vol. XL, p. 42 (1886).

‡ Pearson, Phil. Trans. (A), vol. 187, p. 253 (1896); A, vol. 200, p. 1 (1903). Edgeworth, Phil. Mag., 1892, vol. 34, p. 190 (1903). Yule, Journal of the Royal Statistical Society, vol. 60, p. 812. Cf. Bertrand, Calcul de Probabilitès, Chap. IX, Czuber, Theorie der Beobachtungsfehler, Dritter Teil.

(*b*) The arrays of the correlation table are normal distributions; that is, any section of the surface $z = \psi(x, y)$ by assigning x or y is a normal* (Gaussian) curve.

These conditions, if we make an additional assumption of a purely analytic nature, lead so readily to (13) that we shall not give the details of the argument.

Condition (*b*) implies that $\psi(x, y)$ is of the form $e^{X(x,y)}$, and the additional assumption is that $X(x, y)$ can be expanded in a convergent power series in x and y.

Certain Loci on the Plane of Distribution in Normal Correlation.

9. Ellipses of Equal Probability.—In what follows, it seems to me to add somewhat to the description to regard points (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n) (Figs. 2 and 3) as particles of equal mass on the plane of distribution. The average density of distribution of particles on an area may be defined as the relative frequency† of particles on that area divided by the area. From this definition, we may pass to the limiting case and say that z in the surface

$$z = \psi(x, y)$$

gives the limiting value of the density at any point on the plane of distribution. The curve along which the density of distribution is constant is the ellipse obtained by assigning a given value to z in equation (13), and interpreting the result on the plane of distribution. The infinite system of homothetic ellipses obtained by assigning different values to z plays an important role in Bravais's fundamental memoir.‡ Such an ellipse is sometimes called an ellipse of equal probability.§ We shall deal in this paper (§ 10) with one of these ellipses of special interest. For this purpose, the equation of any ellipse of the system may be written in the form

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2rxy}{\sigma_x\sigma_y} = \lambda^2. \tag{18}$$

The area of this ellipse is

$$\frac{(\pi\lambda^2\sigma_x\sigma_y)}{\sqrt{1 - r^2}}$$

and the semiaxes are given by $a = k\lambda$ (19) and $b = k'\lambda$ (20), where k and

---

* The normal curve in rectangular coördinates is defined by the equation

$$y = e^{ax^2+bx+c},$$

where a is negative. It is easily shown that $a = -1/2\sigma^2$, where σ^2 is defined as the second moment of the area under the curve, about the line $x = -b/a$, divided by the area.

† The "relative frequency" of events and the probability of an event are used interchangeably in this paper.

‡ Loc. cit.

§ Cf. Bertrand, loc. cit.

k' are functions of σ_x, σ_y, and r. The probability that a particle will fall within any ellipse obtained by assigning λ, is given by

$$\frac{2\pi\sigma_x\sigma_y}{2\pi\sigma_x\sigma_y(1-r^2)}\int_0^\lambda e^{-\frac{1}{2(1-r^2)}\lambda^2}\lambda d\lambda = 1 - e^{-\frac{\lambda^2}{2(1-r^2)}}. \quad (21)$$

10. The Ellipse of Maximum Probability.—We shall now determine the ellipse along which, for a given small ring $\Delta\lambda$, we should expect more particles than along any other ellipse of the system.

The perimeter of the ellipse of semiaxes $k\lambda$ and $k'\lambda$ (§ 9) is given by

$$4k\lambda\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\varphi}\,d\varphi,$$

where e^2 is independent of λ. Since the integral is independent of λ, we may write the perimeter of the ellipse in the form $k''\lambda$, where k'' is independent of λ. Hence, the total probability that a particle falls in a small ring between ellipses λ and $\lambda + \Delta\lambda$ is given by

$$k''\lambda e^{-\frac{\lambda^2}{2(1-r^2)}}\Delta\lambda. \quad (22)$$

This expression is a maximum when $\lambda^2 = 1 - r^2$. Hence, what may well be called the ellipse of maximum probability is

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2rxy}{\sigma_x\sigma_y} = 1 - r^2. \quad (23)$$

To illustrate the meaning of this ellipse, in Bertrand's illustration of shooting a thousand shots at a target, the probability is greater that a shot will fall along this ellipse than along any other ellipse of the infinite system.

We may further easily prove the following theorem: *The ellipse of maximum probability* is identical *to the orthogonal projection of parabolic points of the correlation surface on the plane of distribution.*

To prove this theorem, we simply find the locus of parabolic points on the surface (13) by means of the well-known condition

$$\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2.$$

This gives

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2rxy}{\sigma_x\sigma_y} = 1 - r^2,$$

which establishes the theorem.

Attention has often been called to another ellipse known as the "probable" ellipse. The probable ellipse is defined as that ellipse of the system

such that the probability is $\frac{1}{2}$ that a particle falls within it. This means, by (21), that λ is such that

$$e^{-\frac{\lambda^2}{2(1-r^2)}} = \frac{1}{2},$$

or

$$\lambda^2 = 1.3863(1 - r^2). \tag{24}$$

Hence, the probable ellipse is larger than the ellipse of maximum probability. In fact, the probability that a particle falls within the ellipse of maximum probability is $1 - e^{-\frac{1}{2}} = 0.3935$, while that of falling within the probable ellipse is, by definition, $\frac{1}{2}$.

For the illustration of statures of husbands and wives, these two ellipses are shown in Fig. 3. By actual count from the drawing (Fig. 3), it appears

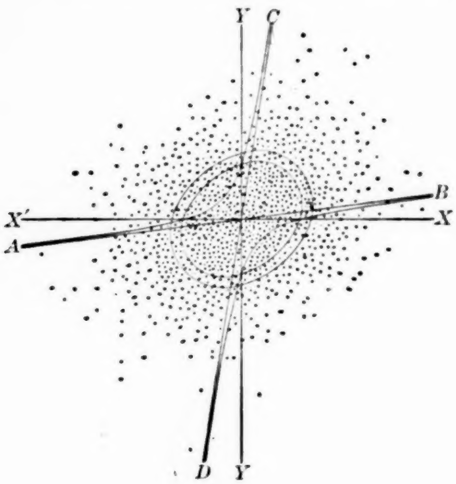

FIG. 3.

that 536 of the 1,078 points are within* the probable ellipse and 412 are within the ellipse of maximum probability. These numbers differ from the theoretical values by amounts well within what should be expected with 1,078 points in all.

11. Separation of the Plane of Distribution by Lines of Regression (Fig. 2).—The lines of regression and some other lines to be defined presently are of such importance in describing the distribution of particles on our plane of distribution that we shall consider the probability that a particle falls into a given compartment of the plane separated from the rest of the plane by these lines.

Let us take the lines

$$y = l_1 x, \tag{25}$$

$$y = l_2 x, \tag{26}$$

* One half the points on the ellipse are counted within it in making this count.

and determine the probability that a particle falls into a compartment made by these lines. The probability is given by the volume under the correlation surface bounded by (25) and (26), interpreted as planes. The probability is, if we make

$$x = \rho \cos \theta,$$

$$y = \rho \sin \theta,$$

$$P = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \int_{\text{arc tan } l_1}^{\text{arc tan } l_2} \int_0^\infty e^{-\frac{\rho^2}{2(1-r^2)}\left(\frac{\cos^2\theta}{\sigma_x^2} + \frac{\sin^2\theta}{\sigma_y^2} - \frac{2r\sin\theta\cos\theta}{\sigma_x\sigma_y}\right)} \rho d\rho d\theta$$

$$= \frac{1}{2\pi}\left(\text{arc tan}\frac{\sigma_x l_2 - r\sigma_y}{\sigma_y\sqrt{1-r^2}} - \text{arc tan}\frac{\sigma_x l_1 - r\sigma_y}{\sigma_y\sqrt{1-r^2}}\right). \quad (27)$$

If (25) and (26) are lines of regression so that

$$l_1 = r\frac{\sigma_y}{\sigma_x}, \quad l_2 = \frac{1}{r}\frac{\sigma_y}{\sigma_x},$$

then

$$P = \frac{1}{2\pi}\text{arc cos } r \qquad (0 \gtreqless \text{arc cos } r \gtreqless \pi). \quad (28)$$

When r is positive, this is the probability that a point will fall into one of the compartments of the smaller angles between the lines of regression.

To find the probability that a particle falls into this same region under independence of X and Y, we make $r = 0$ in (27) before making the substitutions

$$l_1 = r\frac{\sigma_y}{\sigma_x}, \quad l_2 = \frac{1}{r}\frac{\sigma_y}{\sigma_x}.$$

This gives

$$P' = \frac{1}{2\pi}\text{arc cos}\frac{2r}{1+r^2}.$$

Hence, the excess relative frequency in this compartment of the table is

$$\frac{1}{2\pi}\left(\text{arc cos } r - \text{arc cos}\frac{2r}{1+r^2}\right).$$

12. Loci Along Which the Frequency of Particles Bears a Simple Relation to the Frequency under Independence.—If the equality [§ 6, (7)] $\psi(x, y) = f(x)\cdot\varphi(y)$ is not an identity, it may be interpreted as the curve in the plane of distribution along which particles are distributed with the same frequency as they would be under independence. This curve and its separation of the plane of distribution have been treated by Pearson* for the case of normal correlation.

* Drapers' Company Research Memoirs, Biometric Series, I, XIII.

That treatment is easily extended for normal correlation to find the curve along which points are k times as frequent as on the hypothesis of independence. I call k the *intensity of distribution* with respect to independence. In the general case, this curve has the equation

$$\psi(x, y) = kf(x)\,\varphi(y). \tag{30}$$

For normal correlation, (30) takes the form

$$\frac{1}{2\pi\sigma_x\sigma_y}\sqrt{1-r^2}\,e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2rxy}{\sigma_x\sigma_y}\right)} = \frac{k}{2\pi\sigma_x\sigma_y}e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)},$$

which may be easily simplified to

$$\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2xy}{r\sigma_x\sigma_y}=\frac{(1-r^2)}{-r^2}\log(k^2-k^2r^2). \tag{31}$$

This hyperbola divides the plane of distribution into two regions in one of which the intensity of distribution is greater than k, while in the other it is less than k. By giving different values to k, we obtain an infinite system of hyperbolas. The case of

$$k^2 = \frac{1}{1-r^2}$$

is of special interest as it may be regarded as the intensity of distribution at the centroid of particles. Making

$$k = \frac{1}{\sqrt{1-r^2}}$$

in (31) causes the equation to degenerate into the two straight lines

$$y = \frac{1}{r}\frac{\sigma_y}{\sigma_x}x(1 \pm \sqrt{1-r^2}). \tag{32}$$

These lines are shown as lines AB and CD on Fig. 3. The hyperbola along which the frequency is the same as under independence is also shown on the same figure. The probability that a particle will fall into a specified one of the four compartments [made by lines (32)] of the plane of distribution is given by substitution for l_1 and l_2 in (27). This gives for the probability $\frac{1}{4}$. That is, the probability is just the same that a particle belongs to one of the four compartments as to any other.

The probability that a point belongs to the region of one of these compartments, in case of independence, is

$$\frac{1}{2\pi}\text{arc cos } r.$$

13. Separation of Plane by What Would Be Lines of Regression Under Independence.—These lines are simply our coördinate axes. The probability that a particle falls into a specified quadrant under independence is, of course, ¼. The probability that a particle will fall into the first quadrant, in the case of normal correlation, is given by (27) as

$$\frac{1}{4}+\frac{1}{2\pi}\text{arc sin } r \quad \left(-\frac{\pi}{2}<\text{arc sin } r<\frac{\pi}{2}\right).$$

And the probability of falling into the fourth quadrant is

$$\frac{1}{4}-\frac{1}{2\pi}\text{arc sin } r \quad \left(-\frac{\pi}{2}<\text{arc sin } r<\frac{\pi}{2}\right).$$

Hence, we may say, in the case of normal correlation, if we know only in regard to a variate of class X that it is above the mean, that the odds are $\frac{1}{4}+1/2\pi$ arc sin r to $\frac{1}{4}-1/2\pi$ arc sin r that the corresponding variate of class Y is above the mean.

In our example of the correlation of husbands and wives in stature, the numerical values of the odds are 0.2962 to 0.2038 or 3 to 2 approximately that the stature of the wife is above the mean if it is given that the husband is above the mean of husbands in stature.

The study of the separations of the plane of distribution by such lines as those here considered throws considerable light on the character of normally correlated statistical data.

University of Illinois.

Thall

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